WAVE DECAY IN THE ASYMPTOTICALLY FLAT STATIONARY SETTING

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ABSTRACT

Katrina Morgan: Wave decay in the asymptotically flat stationary setting
(Under the direction of Jason Metcalfe)

The current work[1] considers solutions to the wave equation on asymptotically flat, stationary, Lorentzian spacetimes in (1+3) dimensions. We investigate the relationship between the rate at which the geometry tends to flat and the pointwise decay rate of solutions. Tataru 2013 studied the case where the spacetime tends toward flat at a rate of $|x|^{-1}$ and obtained a t^{-3} pointwise decay rate. Here we extend the result to geometries tending toward flat at a rate of $|x|^{-\kappa}$ and establish a pointwise decay rate of $t^{-\kappa-2}$ for $\kappa \in \mathbb{N}$ with $\kappa \geq 2$. A weak local energy decay estimate is assumed to hold, which restricts the geodesic trapping allowed to occur on the underlying geometry. We use the resolvent to connect the time Fourier Transform of a solution to the Cauchy data. The resolvent is initially well-defined in the lower half plane, and once we extend it to the real axis we are able to invert the Fourier Transform to obtain decay. The final decay rate is obtained via analysis of the zero resolvent, whose behavior depends on the rate at which the geometry tends to flat.

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To my parents, Pamela and Richard Morgan.

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CHAPTER 1

Introduction

1.1 Background and Heuristics

This work examines the effect of the far away metric behavior on pointwise wave decay on asymptotically flat, stationary backgrounds in $(1+3)$ dimensions. Roughly, a spacetime geometry is said to be asymptotically flat if the metric coefficients tend toward the flat Minkowski metric ($\mathfrak{m} = \mathrm{diag}(-1,1,1,1)$ in (t,x) coordinates) as $|x| \to \infty$. Our precise definition is given later in the introduction. The background geometry is said to be stationary if the metric coefficients are time independent. The result stated in the main theorem interpolates between two known results for $(1+3)$ dimensional spacetimes with time independent metrics, which are stated in Table 1.1. Here and throughout the paper we take $r := |x|$. The local decay rates given in Table 1.1 hold for compactly supported initial data. In the case of [33] and the current work, the assumptions on the initial data can be weakened.

Since we are working in three spatial dimensions, sharp Huygens' principle says that if the initial data is smooth and compactly supported then solutions to the wave equation at any point in space on flat Minkowski space decay all the way to 0 for sufficiently large t. In Table 1.1 we make a weaker statement that solutions to the wave equation have arbitrarily fast polynomial time decay.

Asymptotically flat spacetimes arise in General Relativity, which has motivated a variety of purely mathematical questions about asymptotically flat geometries and the resulting wave behavior. Tataru's work ([33]), which established a t^{-3} local decay rate for a class of background geometries that tend toward flat at a rate of r^{-1}, was motivated by a conjecture called Price's Law. The

	Metric Behavior	Local Wave decay
[33]	$\mathfrak{g} = flat + \mathcal{O}(r^{-1})$	$\lvert u(t,x)\rvert \lesssim_x t^{-3}$
Current Work:	$\mathfrak{g} = flat + \mathcal{O}(r^{-\kappa})$	$\lvert u(t,x)\rvert \lesssim_x t^{-\kappa-2}$
Sharp Huygens':	$\mathfrak{g} = flat$	$\lvert u(t,x)\rvert \lesssim_x t^{-\infty}$

Table 1.1: Summary of Decay Rates

conjecture was posited by Richard Price in the 1970's in [27]. Price's Law predicted a t^{-3} pointwise local decay rate for waves on the Schwarzschild metric - the geometry describing space in the presence of a single, non-rotating, spherically symmetric black hole.

Several works have explored local decay of waves on different asymptotically flat spacetimes. The question of proving Price's law was explored in [33] and also in [13], where they analyze the wave behavior via spherical modes using the spherical symmetry of the Schwarzschild metric. The Kerr spacetime describes space in the presence of a single, axially symmetric, rotating black hole. Local decay rates for the Kerr spacetime have been studied in [14] and [10]. In [25] the authors proved Price's Law for non-stationary asymptotically flat spacetimes and established the t^{-3} decay rate for a class of pertubations the Kerr spacetime. The techniques in [13], [33], and the current work involve taking the Fourier transform in time and therefore do not readily extend to non-stationary geometries.

Tataru's paper includes an assumption on the local energy decay of solutions to the wave equation on the background geometry instead of considering a specific background such as Schwarzschild. The local energy decay estimate holds if the underlying geometry allows waves to spread out enough so that the energy within compact spatial regions decays sufficiently quickly to be integrable in time. It has been used to establish dispersive estimates such as Strichartz estimates (global, mixed norm estimates) in [24], [34], and [19] and pointwise estimates in [9], [25], [33] (among others). The work of [19] established Strichartz estimates on the Schwarzschild spacetime and showed that a weak form of the local energy decay estimate holds on the Schwarzschild spacetime. Weak local energy decay on the Schwarzschild geometry was also established in [3] and [11]. For the Kerr spacetime with low angular momentum, weak local energy decay estimates have been proven in [11], [2], and [12]. The assumptions in [33] therefore hold for Schwarzschild and Kerr with low angular momentum. A major challenge in obtaining local energy estimates for the Kerr and Schwarzschild geometries is the presence of trapping, in which a portion of the wave flow remains within a fixed set.

A natural question arising from Tataru's result is: What aspects of the geometry dictate Price's Law? There are three locations that are a priori suspected to affect this decay rate: the event horizon, the photon sphere, and the behavior of the perturbation at spatial infinity. The current work shows that the metric behavior at spatial infinity dictates the local decay rate of waves when local energy estimates hold. On the Schwarzschild background, trapping occurs in two areas called the event horizon and the photon sphere. The trapping at the event horizon has been shown to be trivial due to what is known as the red-shift effect, which guarantees energy decay along the trapped rays ([10], [11]). The photon sphere corresponds to a fixed radius, and rays initially tangential to this surface remain there for all time. The behavior on Kerr backgrounds is more complicated. The trapping at the event horizon is similarly known to be trivial, but the other trapped set does not occur on a fixed radius and can only be described in phase space. In order to deal with trapping, a weak local energy estimate with zero coefficients on the trapped set is often introduced. If this holds, then one obtains local energy estimates on the trapped set with a derivative loss. Our definition of the weak local energy decay estimate includes this derivative loss.

Questions similar to the aim of this paper were studied in [5] and [4]. The authors established local decay rates for waves on asymptotically flat, stationary spacetimes which tend toward flat at different rates. One key difference between the works of Bony and Hafner and the current work is that we handle full Lorentzian perturbations of flat Minkowski space. In [5] and [4] the authors considered solutions to the wave equation for perturbations of the Laplacian. The metrics considered in this paper may contain $dtdx_i$ terms, which results in mixed space-time differential operators in our wave operator. Another difference is that we allow for the possibility of unstable trapping on our background. In the work of Bony and Hafner, a nontrapping assumption is used in order to obtain decay for the high frequency part of a solution to the wave equation. The non-trapping assumption is not needed for the low frequency part. We note that [5] considers $(1, n)$ dimensional geometries for $n \geq 2$ and [4] considers n odd with $n \geq 3$. The current work only studies $(1, 3)$ dimensional spacetimes. We also improve upon the decay rates obtained in [5] and [4].

Roughly speaking, the low frequency behavior of a solution to the wave equation is sensitive to the metric behavior at spatial infinity while the high frequency behavior is sensitive to trapping. In the current work, we assume the weak local energy decay estimate holds so that some trapping may occur, but the estimate provides enough information to obtain decay for the high frequency part of our solution u. In fact, we are able to obtain arbitrarily fast decay for the high frequency part of u (as long as the initial data has sufficient regularity). It is the low frequency behavior, which depends on the metric perturbation at spatial infinity, that dictates the local decay rate.

1.2 The Wave Equation

The flat wave operator is given by

$$\Box = -\partial_t^2 + \sum_{i=1}^{3} \partial_{x_i}^2 = -\partial_t^2 + \Delta_x$$

where Δ_x indicates the spatial Laplacian. Throughout the paper we write $\Delta = \Delta_x$ without explicitly specifying we are using the spatial Laplacian. Similarly we write $\nabla = \nabla_x$ for the spatial gradient. When both time and spatial derivatives are considered we use ∂.

The wave operator associated to a Lorentzian metric $\mathfrak{g} = \mathfrak{g}_{\alpha\beta} d\alpha d\beta$ with signature (3,1) is given by

$$\Box_{\mathfrak{g}} = \frac{1}{\sqrt{|\mathfrak{g}|}} \partial_\alpha \sqrt{|\mathfrak{g}|} \mathfrak{g}^{\alpha\beta} \partial_\beta \tag{1.1}$$

where $|\mathfrak{g}|$ is the determinant of the matrix associated to the metric and α and β are summed over both time and space dimensions. Throughout the paper we use Latin indices i, j to indicate only spatial dimensions are being considered and Greek indices α, β to indicate both space and time dimensions are being considered.

The flat metric, known as the Minkowski metric, is given in rectangular coordinates by

$$\mathfrak{m} = -dt^2 + \sum_{i=1}^{3} dx_i^2. \tag{1.2}$$

Here the equation for $\Box_{\mathfrak{g}}$ yields $\Box_{\mathfrak{m}} = \Box$ so that the (1.1) does in fact give the flat wave operator in the case of the flat Minkowski spacetime, as one would expect.

1.3 Energy Estimates

We are interested in the Cauchy problem

$$(\Box_{\mathfrak{g}} + V)u = f, \qquad u(0,x) = u_0, \qquad \partial_t u(0,x) = u_1 \tag{1.3}$$

where V is a scalar potential. The assumptions placed on $\mathfrak{g}$ and V are given in Section 1.6. The Cauchy data at time t is denoted $u[t] = \Big(u(t,\cdot), \partial_t u(t,\cdot)\Big)$.

Definition 1.1. *We say that the evolution* (1.3) *satisfies the uniform energy bounds if:*

$$\|u[t]\|_{\dot{H}^{k,1}\times H^k} \le c_k(\|u[0]\|_{\dot{H}^{k,1}\times H^k} + \|f\|_{L^1H^k}), \qquad t \ge 0, \quad k \ge 0. \tag{1.4}$$

Here H^k denotes the usual Sobolev space, and we say $\phi \in \dot{H}^{k,1}$ if $\nabla\phi \in H^k$.

1.3.1 Local Energy Decay

Local energy decay estimates originated in the work of Morawetz ([26]) where the author established a dispersive estimate for solutions to the flat wave equation. In [16] the authors presented a new approach for proving existence of solutions for nonlinear waves which relied on obtaining a Morawetz-type estimate. The use of local energy estimates has since become a standard tool for studying nonlinear wave equations (e.g. [6], [15], [20], [31], [21], [17], [18], [35], among many others). As stated in the preceding discussion, local energy estimates have also proved to be a powerful tool for establishing other dispersive estimates.

The original Morawetz estimate considered a solution u to the homogeneous flat wave equation with initial data u_0, u_1 and states

$$\int_0^t \int_{\mathbb{R}^3} \frac{1}{|x|}|\not{\nabla} u|^2(t,x)\ dtdx \lesssim \|\nabla u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2.$$

Restricting to compact regions in space, one is able to obtain similar bounds on u and its derivatives (see e.g. [16], [30], and [32]). Our definitions for the local energy norms will restrict to dyadic spatial regions. We use $\langle r\rangle$ to indicate a smooth function of r such that $\langle r\rangle \ge 1$ and $\langle r\rangle = r$ for $r > 2$, and we define $A_m := \{x : 2^m \le \langle r\rangle \le 2^{m+1}\}$. One benefit of using these dyadic regions is that $r \approx 2^m$ on the region of integration, so the weights in the local energy norm can roughly be treated as constant within the region of integration. The precise form of the local energy estimates used in the current work are provided below.

The local energy norm is defined by

$$\|u\|_{LE} = \sup_m \|\langle r\rangle^{-\frac{1}{2}} u\|_{L^2(\mathbb{R}_+\times A_m)}. \tag{1.5}$$

Its H^1 analogue is given by

$$\|u\|_{LE^1} = \|\nabla u\|_{LE} + \|\langle r\rangle^{-1} u\|_{LE} \tag{1.6}$$

and the dual norm is given by

$$\|f\|_{LE^*} = \sum_m \|\langle r\rangle^{\frac{1}{2}} f\|_{L^2(\mathbb{R}_+\times A_m)}. \tag{1.7}$$

For functions with higher regularity we define the following norms

$$\|u\|_{LE^{1,N}} = \sum_{j\le N} \|\partial^j u\|_{LE^1}, \qquad \|f\|_{LE^{*,N}} = \sum_{j\le N} \|\partial^j f\|_{LE^*}. \tag{1.8}$$

The spatial counterparts of the LE and LE^* space-time norms are

$$\|v\|_{\mathcal{LE}} = \sup_m \|\langle r\rangle^{-\frac{1}{2}} v\|_{L^2(A_m)}; \qquad \|g\|_{\mathcal{LE}^*} = \sum_m \|\langle r\rangle^{\frac{1}{2}} g\|_{L^2(A_m)} \tag{1.9}$$

with the higher regularity norms defined by

$$\|v\|_{\mathcal{LE}^N} = \sum_{j\le N} \|\nabla^j v\|_{\mathcal{LE}}, \qquad \|g\|_{\mathcal{LE}^{*,N}} = \sum_{j\le N} \|\nabla^j g\|_{\mathcal{LE}^*}. \tag{1.10}$$

Definition 1.2. *We say that the evolution* (1.3) *satisfies the local energy decay estimate if:*

$$\|u\|_{LE^{1,N}} \le c_N(\|u[0]\|_{H^{N,1}\times H^N} + \|f\|_{LE^{*,N}}), \qquad N \ge 0 \tag{1.11}$$

in $\mathbb{R}\times\mathbb{R}^3$.

The local energy decay estimate is known to hold in several nontrapping geometries. For sufficiently small perturbations of flat space without trapping, local energy decay was established in [1], [21], and [23]. The case of stationary product manifolds was considered in [8], [5], and [31]. The nontrapping case was studied more generally in [22]. If trapping occurs then the local energy decay estimate does not hold ([28], [29]).

1.3.2 Weak Local Energy Decay

Trapping on the background geometry may be stable or unstable. A spacetime with trapping where every trapped geodesic is unstable may still admit a weaker form of the local energy decay estimate. In the case of trapping, there is necessarily a loss of derivatives on the right hand side of the estimate (see e.g. [7]).

Definition 1.3. *We say that the evolution has the weak local energy decay property (or satisfies the weak local energy decay estimate) if:*

$$\|u\|_{LE^{1,N}} \leq c_N(\|u[0]\|_{\dot{H}^{N+3,1}\times H^{N+3}} + \|f\|_{LE^{*,N+3}}), \qquad N \geq 0. \tag{1.12}$$

As discussed in Section 1.1, the weak local energy decay estimate is known to hold on the Schwarzschild spacetime and Kerr with low angular momentum.

1.4 Vector Fields and Weighted Sobolev Spaces

Our argument will use vector field methods. In this section we set the notation for the relevant vector fields.

- Rotations: $\Omega = \{\Omega_{ab} \mid a, b = 1, 2, 3\}$

$$\Omega_{ab} = x_a\partial_b - x_b\partial_a$$

- Translations: $T = \{T_a \mid a = 1, 2, 3\}$

$$T_a = \nabla_a$$

- Scaling: $S = S_r - S_\tau$

$$S_r = r\partial_r \quad S_\tau = \tau\partial_\tau.$$

Note that the scaling vector field we use is taken in time frequency space and therefore differs from the scaling vector field in physical space which is given by $r\partial_r + t\partial_t$. This is because we use the vector field arguments only on the time Fourier transform side. We denote the collection of all such vector fields by $\Gamma = \{\Omega, T, S\}$. We write $\Gamma^{<n}$ to denote a linear combination of Γ^α for $|\alpha| < n$: $\Gamma^{<n} := \sum_{|\alpha|<n} c_\alpha \Gamma^\alpha$.

We use the vector fields to define a weighted Sobolev type norm. The initial data will be assumed to lie in such a space. The weighted Sobolev spaces $Z^{n,q}$ are defined by

$$\|\phi\|_{Z^{n,q}} = \sup_{i+j+k\leq n} \|\langle r\rangle^q T^i \Omega^j S_r^k \phi\|_{\mathcal{LE}^*}. \tag{1.13}$$

1.5 Symbol Classes

We will assume that the metric coefficients belong to certain symbol classes. In this section we define the relevant notation.

The symbol classes $S(r^q), \ell^1 S(r^q), S(\log r)$ are defined as follows:

$$\phi(x) \in S(r^q) \Leftrightarrow \|\langle r\rangle^{j-q}\partial^j f(x)\|_{L^\infty(\mathbb{R}^3)} \lesssim_j 1 \qquad j \in \{0,1,2,\dots\}$$
$$\phi(x) \in \ell^1 S(r^q) \Leftrightarrow \sum_m 2^{m(j-q)}\|\partial^j f(x)\|_{L^\infty(A_m)} \lesssim_j 1 \qquad j \in \{0,1,2,\dots\}$$
$$\phi(x) \in S(\log r) \Leftrightarrow \|(\log\langle r\rangle)^{-1} f(x)\|_{L^\infty(\mathbb{R}^3)} \lesssim 1 \text{ and } \|\langle r\rangle^j\partial^j f(x)\|_{L^\infty(\mathbb{R}^3)} \lesssim_j 1, \qquad j \in \{1,2,3,\dots\}.$$

If $\phi \in S(r^q)$ is radial, we write $\phi \in S_{rad}(r^q)$. We indicate radial functions in the other symbol classes analogously.

In some of our calculations we use the notation ρ^q_ℓ to indicate a representative of the symbol class $\ell^1 S(r^q)$. Similarly, we use ρ^q to represent $S(r^q)$ and ρ^q_r to represent $S_{rad}(r^q)$.

1.6 Statement of Main Theorem

We consider a Lorentzian metric $\mathfrak{g}$ with the following properties:

1. $\mathfrak{g}$ is stationary, meaning the metric coefficients are time independent.

2. The submanifolds $t = constant$ are space-like.

3. Let $\kappa \in \mathbb{N}$ with $\kappa \geq 2$. $\mathfrak{g}$ is asymptotically flat in the sense that the metric $\mathfrak{g}$ can be written as

$$\mathfrak{g} = \mathfrak{m} + \mathfrak{f} + \mathfrak{h}$$

where

$$\mathfrak{f} = \mathfrak{f}_{00}(x)dt^2 + \mathfrak{f}_{0i}(x)dtdx_i + \mathfrak{f}_{ij}(x)dx_idx_j$$

with $\mathfrak{f}_{\alpha\beta} \in \ell^1 S(r^{-\kappa})$ for $\alpha, \beta \in \{0,1,2,3\}$ and

$$\mathfrak{h} = \mathfrak{h}_{tt}(r)dt^2 + \mathfrak{h}_{tr}(r)dtdr + \mathfrak{h}_{rr}(r)dr^2 + \mathfrak{h}_{\omega\omega}(r)r^2d\omega^2$$

with $\mathfrak{h}_{\gamma\delta} \in S_{rad}(r^{-\kappa})$ for $\gamma, \delta \in \{t, r, \omega\}$. Here $d\omega^2 = d\theta^2 + \sin^2\theta d\phi^2$.

Note that $\mathfrak{h}$ is spherically symmetric with radial coefficients that exhibit less decay than the coefficients of $\mathfrak{f}$, where no spherical symmetry is assumed.

Theorem 1.4. *Let $\mathfrak{g}$ be a (1+3)-dimensional spacetime satisfying the metric assumptions above. Let V be a potential of the form*

$$V(x) = V_\ell(x) + V_r(r), \qquad V_\ell \in \ell^1 S(r^{-\kappa-2}), \quad V_r \in S_{rad}(r^{-\kappa-2}). \tag{1.14}$$

Assume the homogeneous Cauchy problem

$$(\Box_g + V)u(t,x) = 0, \quad u(0,x) = u_0, \quad \partial_t u(0,x) = u_1 \tag{1.15}$$

satisfies the uniform energy bound (1.4) *and the weak local energy decay assumption* (1.12)*. If u solves* (1.15) *with $u_0 \in Z^{\nu+1,\kappa}$ and $u_1 \in Z^{\nu,\kappa+1}$ for $\nu \geq 31\kappa + 168$, then in normalized coordinates (see Chapter 3 for details) u satisfies the bounds*

$$|u(t,x)| \lesssim \frac{1}{\langle t\rangle\langle t-|x|\rangle^{\kappa+1}}\big(\|u_0\|_{Z^{\nu+1,\kappa}} + \|u_1\|_{Z^{\nu,\kappa+1}}\big) \tag{1.16}$$

$$|\partial_t u(t,x)| \lesssim \frac{1}{\langle t\rangle\langle t-|x|\rangle^{\kappa+2}}\big(\|u_0\|_{Z^{\nu+1,\kappa}} + \|u_1\|_{Z^{\nu,\kappa+1}}\big). \tag{1.17}$$

1.7 Argument Summary

Our argument relies on fixing a coordinate system that allows us to write the operator $(\Box_g + V)$ in the form

$$P = -\partial_t^2 + \Delta + \partial_t P^1 + P^2 \tag{1.18}$$

where P^1 and P^2 are spatial operators of order 1 and 2, respectively. The coefficients of the operators depend on the metric coefficients assumed in the main theorem. We then use the resolvent (denoted R_τ) to connect the time Fourier transform of a solution u to the Cauchy problem (1.15) with the initial data.

We define the resolvent to be the inverse of the image of P under the time Fourier transform, when the inverse exists. We note in the classical definition, the resolvent of an operator Q is given by $(Q-\lambda)^{-1}$ when the inverse exists. If $Q = \Delta$, then the classical resolvent is a useful tool for studying the flat wave equation, $\Box u = (-\partial_t^2 + \Delta)u = 0$. If $\Box u = 0$ and $u(0, \cdot) = 0$ then taking the Fourier Transform in time of $\Box u$ and integrating by parts yields $\hat{u}(\tau, x) = (\Delta + \tau^2)^{-1}\partial_t u(0, x)$. Thus the time Fourier Transform of the solution is given by applying the resolvent of the Laplacian (with $\lambda = -\tau^2$) to the initial data. Our definition of the resolvent associated to a wave operator P allows us to take advantage of the connection with the Fourier Transform just described. By our definition, the resolvent associated to the operator $\Box = -\partial_t^2 + \Delta$ is given by $(\Delta + \tau^2)^{-1}$. More generally, the resolvent associated to an operator P of the form (1.18) can be viewed as a perturbation of the classical resolvent of the Laplacian, where the perturbation depends on the spectral parameter τ.

We will establish that if u solves (1.15) then

$$\hat{u}(\tau) = R_\tau(-i\tau u_0 + P^1 u_0 - u_1). \tag{1.19}$$

The final pointwise decay rate is then proved by analyzing the resolvent and inverting the Fourier transform. Our argument will be different for high frequencies ($|\tau| \gtrsim 1$) and low frequencies ($|\tau| \lesssim 1$). In the high frequency case we are able to use bounds resulting from the weak local energy decay assumption to obtain arbitrarily fast pointwise decay when inverting the Fourier transform, regardless of the rate at which the background geometry tends toward flat (as long as the initial data exhibits enough regularity). In the low frequency case, we obtain an expansion in powers of r^{-1} for the resolvent at zero frequency and use this to calculate the error in the estimate $R_\tau u_0 \approx (R_0 u_0)e^{-i\tau\langle r\rangle}$ for the resolvent at low frequencies. We then apply the inverse Fourier transform to the terms arising in this estimate. The behavior of these terms dictates the final pointwise decay rate.

This approach is due to [33]. A key difference in our analysis as compared to Tataru's is that we need to go further down in the expansion of the zero resolvent in order to obtain improved decay rates. Changing the expansion then affects the error in the estimate for $R_\tau u_0$ when $|\tau|$ is small. The rate at which the background geometry tends toward flat (indicated by the parameter κ in the statement of the main theorem) ultimately determines how far down in the expansion of the zero resolvent we are able to go, which determines the error terms in our low frequency resolvent estimate and in turn determines the result of inverting the Fourier transform.

In Chapter 2 we present general results on the behavior of the resolvent of operators of the form (1.18). We separate out the resolvent analysis at the beginning of the paper to emphasize the minimal assumptions on P needed to obtain the results. In particular, we note the weak local energy decay assumption allows us to obtain L^2 based bounds on the resolvent associated to an operator P of the form (1.18) with only minimal assumptions on the P^2 piece of the operator (the P^2 operator must have decaying coefficients). In Chapter 3 we fix a normalized coordinate system that allows us to write $(\Box_g + V)$ as in (1.18) with coefficients that are determined by the metric coefficients. The operator P, which we obtain in Chapter 3, satisfies the assumptions for all of the results in Chapter 2 for $\kappa \geq 1$. In Chapter 4 we analyze the resolvent near zero frequency. As stated above, it is this analysis which changes depending on the rate at which the background geometry tends toward flat. In Chapter 5 we use the L^2 based resolvent bounds, which we obtained using the weak local energy estimate along with Sobolev embeddings, to obtain pointwise resolvent bounds. In Chapter 6 we prove the main theorem by inverting the Fourier transform of (1.19).

1.8 Cutoff and Bump Functions

We will use cutoff functions which localize to a given region and cutoff functions which restrict to values greater or less than a given quantity.

The function $\chi_{<1}(r)$ is defined to be a smooth function which is 1 for $r \leq 1$ and 0 for $r \geq 2$. We define $\chi_{>1}(r) := 1 - \chi_{<1}(r)$ so that $\chi_{>1}(r)$ is a smooth function which is 1 for $r \geq 2$ and 0 for $r \leq 1$. We define $\chi_{\approx 1}(r)$ to be a smooth function which is 1 for $1 \leq r \leq 2$ and 0 for $r < 2$ and $r > 4$.

We define $\beta_{\approx m}(r)$ for $m \geq 0$ to be a smooth partition of unity which is subordinate to the dyadic intervals A_m.

When restricting r by dyadic regions, we use $\chi_{\approx m}$ to indicate $\chi_{\approx m}(r) = \chi_{\approx}(\frac{r}{2^m})$, $\chi_{>m}(r) = \chi_{>1}(\frac{r}{2^m})$, and $\chi_{<m}(r) = \chi_{<1}(\frac{r}{2^m})$. In all other contexts we write $\chi_{\approx a} = \chi_{\approx}(\frac{r}{a})$, etc..

We note $\partial_r \chi_{<m}(r)$ and $\partial_r \chi_{>m}(r)$ are each supported on $\langle r \rangle \cong 2^m$ so that

$$\|\partial_r \chi_{<m}(r)\|_{L^2(A_m)} \cong \|\partial_r \chi_{>m}(r)\|_{L^2(A_m)} \cong 2^{\frac{3m}{2}}.$$

CHAPTER 2

General results on the resolvent

In this chapter we collect previously known results on the resolvent associated to an operator P of the form

$$P = -\partial_t^2 + \Delta + \partial_t P^1 + P^2 \tag{2.1}$$

where P^1 and P^2 are first and second order spatial operators, respectively. In the next chapter we will show there exists a coordinate system so that the operator $\Box_{\mathfrak{g}} + V$ in the statement of the main theorem can be replaced by an operator of this form where the coefficients of P^1 and P^2 depend on the coefficients of the metric $\mathfrak{g}$.

We present these general results on the resolvent first to emphasize that the rate at which the background geometry tends toward flat does not yet come into play. We will see there is a minimum rate at which the geometry must tend toward flat for these results to hold, but past this threshold they remain the same. The results in this chapter will depend on whether the uniform energy estimate (1.4) and the weak local energy decay estimate (1.12) hold for the Cauchy problem

$$Pu = f, \qquad u(0,\cdot) = u_0, \qquad \partial_t u(0,\cdot) = u_1. \tag{2.2}$$

The propositions in this chapter are established in [33]. Propositions 2.3, 2.4, and 2.5 are formulated differently than the corresponding propositions in [33] to state the minimal assumptions required of the operator P to obtain the result. Proposition 2.6, Corollary 2.7, and Proposition 2.8 are stated as they appear in [33].

2.1 Defining the Resolvent

We define the operator P_τ as the image of the operator P under the time Fourier Transform, which turns time derivatives into multiplication by $i\tau$. The resolvent associated to P is then defined as the inverse of this operator.

Definition 2.1. *Let P be as in* (2.1)*. We define the operator P_τ associated to P by $\partial_t \mapsto i\tau$ so that*

$$P_\tau := \tau^2 + \Delta + i\tau P^1 + P^2.$$

Definition 2.2. *The resolvent associated to P is denoted R_τ^P (or simply R_τ once P is given) and defined by*

$$R_\tau^P := P_\tau^{-1}$$

when it exists.

Using these definitions we obtain the following result:

Proposition 2.3. *Let P be as in* (2.1) *and assume* (2.2) *satisfies the uniform energy bounds* (1.4)*. If $\Im\tau < 0$, the operator $P_\tau : H^2 \to L^2$ is one-to-one and the range of P_τ is dense in L^2. Furthermore, if u satisfies*

$$Pu = f, \quad u(0,\cdot) = u_0, \quad \partial_t u(0,\cdot) = u_1$$

then

$$\hat{u}(\tau, x) = R_\tau^P(\hat{f}(\tau) - i\tau u_0 + P^1 u_0 - u_1) \tag{2.3}$$

for $\Im\tau < 0$.

Proof. Let Q_τ be a family of τ dependent operators which are defined by

$$Q_\tau g = \hat{u}(\tau)$$

where $u(t,x)$ solves the homogeneous Cauchy problem

$$Pu = 0, \quad u(0) = 0, \quad \partial_t u(0) = -g \in L^2.$$

We will show Q_τ satisfies $Q_\tau P_\tau g = g$ and $P_\tau Q_\tau g = g$, so P_τ is invertible, for $\Im\tau < 0$ and $R_\tau = Q_\tau$. We begin by establishing L^2 based bounds on $Q_\tau g$.

By assumption, the evolution (2.2) satisfies the uniform energy bounds (1.4), which translate into L^2 based bounds for $Q_\tau g$. Using the notation $\|\phi\|_{\dot{H}^{N,1}} = \|\nabla\phi\|_{H^N}$ and setting $u(t,x) \equiv 0$ for

$t < 0$ we find for all $N \geq 0$

$$\begin{aligned}
\|Q_\tau g\|_{\dot{H}^{N,1}} &= \sum_{j\leq N} \left(\int_{\mathbb{R}^3} \left| \int_0^\infty \nabla^j \nabla e^{-it\tau} u(t,x)\ dt \right|^2\ dx \right)^{1/2} \\
&\leq \sum_{j\leq N} \int_0^\infty \left(\int_{\mathbb{R}^3} \left| e^{-it\tau} \nabla^{j+1} u(t,x) \right|^2\ dx \right)^{1/2} dt \quad \text{by the Minkowski Integral Inequality} \\
&= \int_0^\infty e^{t\Im\tau} \|u(t,\cdot)\|_{\dot{H}^{N,1}(\mathbb{R}^3)}\ dt \\
&\lesssim \|g\|_{H^N} \int_0^\infty e^{t\Im\tau}\ dt \\
&= \frac{1}{|\Im\tau|} \|g\|_{H^N}.
\end{aligned}$$

Similarly we calculate

$$\begin{aligned}
|\tau| \|Q_\tau g\|_{H^N} &= \sum_{j\leq N} \left(\int_{\mathbb{R}^3} \left| \int_0^\infty e^{-it\tau} \nabla^N \partial_t u(t,x)\ dt \right|^2\ dx \right)^{1/2} \\
&\leq \int_0^\infty e^{t\Im\tau} \|\partial_t u(t,\cdot)\|_{H^N(\mathbb{R}^3)}\ dt \\
&\lesssim \frac{1}{|\Im\tau|} \|g\|_{H^N}.
\end{aligned}$$

Thus for all $N \geq 0$, we have

$$\|Q_\tau g\|_{\dot{H}^{N,1}} + |\tau| \|Q_\tau g\|_{H^N} \lesssim \frac{1}{|\Im\tau|} \|g\|_{H^N}. \tag{2.4}$$

So if $g \in H^N$ then $Q_\tau g \in H^{N+1}$ for $\Im\tau < 0$.

In general taking the time Fourier transform of Pu (again setting $u(t,x) \equiv 0$ for $t < 0$) and integrating by parts yields

$$\begin{aligned}
\int e^{-it\tau} Pu\ dt &= -\int (\partial_t^2 u) e^{-it\tau}\ dt + \int (\partial_t P^1 u) e^{-it\tau}\ dt + (\Delta + P^2)\hat{u}(\tau) \\
&= -\left(-\partial_t u(0) + i\tau(-u(0) + i\tau\hat{u}(\tau))\right) - P^1 u(0) + i\tau P^1 \hat{u}(\tau) + (\Delta + P^2)\hat{u}(\tau) \\
&= P_\tau \hat{u}(\tau) + \partial_t u(0) + i\tau u(0) - P^1 u(0)
\end{aligned}$$

so that

$$P_\tau \hat{u}(\tau) = (Pu)\hat{\ } - i\tau u(0) + P^1 u(0) - \partial_t u(0). \tag{2.5}$$

Given $g \in H^1$, we have $Q_\tau g \in H^2$ by (2.4) so $Q_\tau g$ is in the domain of P_τ. Applying the above calculations to our definition of $Q_\tau g$ (where $Pu = 0$, $u(0) = 0$, and $\partial_t u(0) = -g$ and $Q_\tau g = \hat{u}(\tau)$), we find

$$P_\tau Q_\tau g = P_\tau \hat{u}(\tau) = g.$$

It follows that H^1 is contained in the range of $P_\tau : H^2 \to L^2$, so the range is in fact dense in L^2.

Next we aim to show $Q_\tau P_\tau g = g$ and thus establish that P_τ is invertible with $Q_\tau = R_\tau$. To this end, we claim that if $u(t,x)$ solves the nonhomogeneous Cauchy problem

$$Pu = f, \quad u(0,\cdot) = u_0, \quad \partial_t u(0,\cdot) = u_1$$

then

$$\hat{u}(\tau) = Q_\tau(\hat{f}(\tau) - i\tau u_0 + P^1 u_0 - u_1). \tag{2.6}$$

Once (2.6) is established, we can show $Q_\tau P_\tau g = g$. Indeed, assume (2.6) holds and let $g(x) \in H^2$ be given. In order to show that $Q_\tau P_\tau g = g$, we define $u(t,x) = g(x)\mathbf{1}_{t\geq 0}$, where $\mathbf{1}_{t\geq 0}$ is an indicator function that is 1 for $t \geq 0$ and 0 otherwise. Taking the time Fourier transform of $u(t,x)$ yields $\hat{u}(\tau) = \frac{1}{|\Im\tau|}g$. By (2.5) we have

$$\frac{1}{|\Im\tau|}P_\tau g = P_\tau \hat{u} = (Pu)\hat{\ } - i\tau u(0) + P^1 u(0) - \partial_t u(0).$$

Then applying (2.6) gives

$$\begin{aligned} Q_\tau P_\tau g &= |\Im\tau| Q_\tau\Big((Pu)\hat{\ } - i\tau u(0) + P^1 u(0) - \partial_t u(0)\Big) \\ &= |\Im\tau|\hat{u}(\tau) \\ &= g \end{aligned}$$

as desired.

It is left to show (2.6). To do this we use Duhamel's formula and find

$$u(t,x) = \int_0^t u_1(t-s,x;s)\ ds + \partial_t u_2 + u_3 + u_4, \tag{2.7}$$

where $u_1(t,x;s), u_2(t,x), u_3(t,x)$, and $u_4(t,x)$ solve the following:

$$\begin{array}{lll} Pu_1 = 0 & u_1(0,x;s) = 0 & \partial_t u_1(0,x;s) = -f(s,x) \\ Pu_2 = 0 & u_2(0,x) = 0 & \partial_t u_2(0,x) = u_0 \\ Pu_3 = 0 & u_3(0,x) = 0 & \partial_t u_3(0,x) = -P^1 u_0 \\ Pu_4 = 0 & u_4(0,x) = 0 & \partial_t u_4(0,x) = u_1. \end{array}$$

We calculate $\hat{u}(\tau)$ by taking the time Fourier transform of each term in (2.7). For the first term we have

$$\begin{aligned} \int_0^\infty e^{-it\tau} \int_0^t u_1(t-s,x;s)\ dsdt &= \int_0^\infty \int_s^\infty e^{-it\tau} u_1(t-s,x;s)\ dtds \\ &= \int_0^\infty \int_0^\infty e^{-i(t+s)\tau} u_1(t,x;s)\ dtds \\ &= \int_0^\infty e^{-it\tau} \int_0^\infty e^{-is\tau} u_1(t,x;s)\ dsdt. \end{aligned}$$

Set $\beta(t,x;\tau) = \int_0^\infty e^{-is\tau} u_1(t,x;s)\ ds$. Then $\beta(t,x;\tau)$ satisfies

$$P\beta = 0, \qquad \beta(0,x;\tau) = 0, \qquad \partial_t \beta(0,x;\tau) = -\hat{f}(\tau,x)$$

and thus $\hat{\beta}(\tau,x;\tau) = Q_\tau \hat{f}(\tau)$. Applying the time Fourier transform to the remaining terms in (2.7) yields

$$\hat{u}(\tau) = \hat{\beta}(\tau,x;\tau) + i\tau\hat{u}_2(\tau) + \hat{u}_3(\tau) + \hat{u}_4(\tau) = Q_\tau(\hat{f}(\tau) - i\tau u_0 + P^1 u_0 - u_1)$$

as desired. This concludes the proof of (2.6) and thus the proof of the proposition. □

2.2 The Resolvent and Weak Local Energy Decay

In the previous section we showed that when the uniform energy bounds (1.4) hold for (2.2) the resolvent satisfies the estimate (2.4) when $\Im\tau < 0$. If the weak local energy decay estimate (1.12) also holds, then we are able to obtain L^2-based resolvent bounds which are stronger than (2.4) and are uniform as $\Im\tau \to 0$.

The $\mathcal{LE}_\tau$ norm, in which we measure the resolvent $v = R_\tau g$, is defined by

$$\|v\|_{\mathcal{LE}^N_\tau} = \|(|\tau| + \langle r\rangle^{-1})v\|_{\mathcal{LE}^N} + \|\nabla v\|_{\mathcal{LE}^N} + \|(|\tau| + \langle r\rangle^{-1})^{-1}\nabla^2 v\|_{\mathcal{LE}^N}. \tag{2.8}$$

Proposition 2.4. *Let P be as in* (2.1)*. Assume* (2.2) *satisfies the uniform energy bounds* (1.4) *and the weak local energy estimate* (1.12)*. If $\Im\tau < 0$ and $g \in \mathcal{LE}^{*,N+3}$ for fixed $N \in \mathbb{N}$, then $v = R^P_\tau g$ satisfies*

$$\|(\langle r\rangle^{-1} + |\tau|)v\|_{\mathcal{LE}^N} + \|\nabla v\|_{\mathcal{LE}^N} \lesssim \|g\|_{\mathcal{LE}^{*,N+3}}. \tag{2.9}$$

If in addition P^2 satisfies

$$\begin{aligned} P^2 =&\partial_i p_2^{ij}(x)\partial_j + p_2^\omega(r)\Delta_\omega + V \\ &p_2^{ij} \in \ell^1 S(1); \quad p_2^\omega \in \ell^1 S(r^{-2}); \quad V \in S(r^{-2}), \end{aligned} \tag{2.10}$$

then

$$\|v\|_{\mathcal{LE}^N_\tau} \lesssim \|g\|_{\mathcal{LE}^{*,N+4}}. \tag{2.11}$$

Proof. We begin by using the weak local energy decay estimate to establish

$$\|(\langle r\rangle^{-1} + |\tau|)v\|_{\mathcal{LE}^N} + \|\nabla v\|_{\mathcal{LE}^N} \lesssim \sum_{j\le N+3} (1+|\tau|)^{N+3-j}\|\nabla^j g\|_{\mathcal{LE}^*}, \tag{2.12}$$

which we will then be able to use to obtain (2.9). Note the estimate (2.9) bounds the first two terms in the $\mathcal{LE}^N_\tau$ norm. Once we have established (2.9), we therefore only need to bound the last term in the $\mathcal{LE}^N_\tau$ norm to obtain (2.11).

Take $u = e^{it\tau}v$. Then u solves

$$Pu = (P_\tau v)e^{it\tau} = ge^{it\tau}, \quad u(-T) = e^{-iT\tau}v, \quad \partial_t u(-T) = i\tau e^{-iT\tau}v.$$

The weak local energy decay estimate (1.12) applied to the time interval $[-T, 0]$ states that

$$\|u\|_{LE^{1,N}[-T,0]} \lesssim \|\partial u(-T)\|_{H^{N+3}} + \|ge^{i\tau t}\|_{LE^{*,N+3}[-T,0]}. \tag{2.13}$$

Here the notation $LE^{1,N}[-T,0], LE^{*,N}[-T,0]$ indicates we are using the norm defined in the introduction with the L^2 norm taken over $[-T,0]\times A_m$ rather than $\mathbb{R}_+\times A_m$.

We note

$$|\partial u(-T)| = e^{T\Im\tau}|(\tau+\nabla)v|$$

so $\|\partial u(-T)\|_{H^{N+3}}\to 0$ as $T\to\infty$. Using the definitions of the local energy norms, (2.13) thus becomes

$$\begin{aligned}\sum_{j\le N}\sup_{m\ge 0} 2^{-\frac{m}{2}}\left(\|\partial^{j+1}u\|_{L^2([-\infty,0]\times A_m)} + 2^{-m}\|\partial^j u\|_{L^2([-\infty,0]\times A_m)}\right)\\ \lesssim \sum_{j\le N+3}\sum_{m\ge 0} 2^{\frac{m}{2}}\|\partial^j(ge^{it\tau})\|_{L^2([-\infty,0]\times A_m)}.\end{aligned} \tag{2.14}$$

Furthermore we see

$$\|\partial^j u\|^2_{L^2([-T,0]\times A_m)} \approx \sum_{\ell\le j}\|\partial_t^\ell\nabla^{j-\ell}u\|^2_{L^2([-T,0]\times A_m)}$$

and as $T\to\infty$ we have

$$\|\partial_t^\ell\nabla^{j-\ell}u\|_{L^2([-T,0]\times A_m)} = \||\tau|^\ell e^{it\tau}\nabla^{j-\ell}v\|_{L^2([-T,0]\times A_m)} \to \frac{|\tau|^\ell}{\sqrt{2|\Im\tau|}}\|\nabla^{j-\ell}v\|_{L^2(A_m)}.$$

The same calculations yield

$$\|\partial^j(ge^{it\tau})\|_{L^2(A_m\times[-T,0])} \to \sum_{\ell\le j}\frac{|\tau|^\ell}{\sqrt{2|\Im\tau|}}\|\nabla^{j-\ell}g\|_{L^2(A_m)}$$

as $T\to\infty$. Therefore (2.14) yields

$$\begin{aligned}\sup_m 2^{-\frac{m}{2}}\Big(\sum_{j\le N}\sum_{\ell\le j+1}|\tau|^\ell\|\nabla^{j+1-\ell}v\|_{L^2(A_m)} + 2^{-m}\sum_{j\le N}\sum_{\ell\le j}|\tau|^\ell\|\nabla^{j-\ell}v\|_{L^2(A_m)}\Big)\\ \lesssim \sum_m 2^{\frac{m}{2}}\sum_{j\le N+3}\sum_{\ell\le j}|\tau|^\ell\|\nabla^{j-\ell}g\|_{L^2(A_m)}.\end{aligned} \tag{2.15}$$

To handle the double sums we find for any ϕ,

$$\sum_{j\le N}\sum_{\ell\le j}|\tau|^\ell\|\nabla^{j-\ell}\phi\|_{L^2(A_m)} \approx \sum_{j\le N}(1+|\tau|)^{N-j}\|\nabla^j\phi\|_{L^2(A_m)}. \tag{2.16}$$

For the first term on the left hand side of (2.15) we use (2.16) to find

$$\begin{aligned}\sum_{j\le N}\sum_{\ell\le j+1}|\tau|^\ell\|\nabla^{j+1-\ell}v\|_{L^2(A_m)} &= \sum_{j\le N}\sum_{1\le\ell\le j+1}|\tau||\tau|^{\ell-1}\|\nabla^{j-(\ell-1)}\|_{L^2(A_m)}+\sum_{j\le N}\|\nabla^{j+1}v\|_{L^2(A_m)}\\ &\approx\sum_{j\le N}\Big(|\tau|(1+|\tau|)^{N-j}\|\nabla^j v\|_{L^2(A_m)}+\|\nabla^{j+1}v\|_{L^2(A_m)}\Big).\end{aligned}\tag{2.17}$$

Combining (2.16) and (2.17) with (2.15), we obtain

$$\begin{aligned}\sum_{j\le N}\sup_m 2^{-\frac{m}{2}}\Big(|\tau|(1+|\tau|)^{N-j}\|\nabla^j v\|_{L^2(A_m)}+\|\nabla^{j+1}v\|_{L^2(A_m)}+2^{-m}(1+|\tau|)^{N-j}\|\nabla^j v\|_{L^2(A_m)}\Big)&\\ \lesssim\sum_{j\le N+3}\sum_m 2^{\frac{m}{2}}(1+|\tau|)^{N+3-j}\|\nabla^j g\|_{L^2(A_m)}.&\end{aligned}\tag{2.18}$$

Since $1\lesssim(1+|\tau|)$, we see that $\|(\langle r\rangle^{-1}+|\tau|)v\|_{\mathcal{LE}^N}+\|\nabla v\|_{\mathcal{LE}^N}$ is controlled by the left hand side of (2.18) so the proof of (2.12) is complete.

Next we show that (2.9) follows from (2.12). If $|\tau|\lesssim 1$ then (2.12) yields

$$\|(\langle r\rangle^{-1}+|\tau|)v\|_{\mathcal{LE}^N}+\|\nabla v\|_{\mathcal{LE}^N}\lesssim\|g\|_{\mathcal{LE}^{*,N+3}}$$

as desired.

If $|\tau|\gtrsim 1$, we have more work to do in order to remove the τ factors from the right hand side of (2.12). We begin by splitting g into low and high frequency parts

$$g_{high}\cong\int e^{ix\cdot\xi}\hat{g}(\xi)\chi_{>|\tau|}(\xi)\,d\xi,\qquad g_{low}\cong\int e^{ix\cdot\xi}\hat{g}(\xi)\chi_{<|\tau|}(\xi)\,d\xi.$$

Here $\chi_{<|\tau|}(\xi)=\chi_{<1}\Big(\frac{\xi}{|\tau|}\Big)$ and $\chi_{>|\tau|}(\xi)=\chi_{>1}\Big(\frac{\xi}{|\tau|}\Big)$, so $g=g_{low}+g_{high}$. Defining

$$v_{low}=R_\tau g_{low},\qquad v_{high}=R_\tau g_{high}$$

we have $v = v_{low} + v_{high}$. Thus to prove (2.9), it suffices to show

$$\|(\langle r\rangle^{-1} + |\tau|)v_{low}\|_{\mathcal{LE}^N} + \|\nabla v_{low}\|_{\mathcal{LE}^N} \lesssim \|g\|_{\mathcal{LE}^{*,N+3}} \tag{2.19}$$

and

$$\|(\langle r\rangle^{-1} + |\tau|)v_{high}\|_{\mathcal{LE}^N} + \|\nabla v_{high}\|_{\mathcal{LE}^N} \lesssim \|g\|_{\mathcal{LE}^{*,N+3}}. \tag{2.20}$$

For the high frequency part we use (2.12) and the fact that we are in the case $1 \lesssim |\tau|$ to find

$$\begin{aligned}\|(\langle r\rangle^{-1} + |\tau|)v_{high}\|_{\mathcal{LE}^N} + \|\nabla v_{high}\|_{\mathcal{LE}^N} &\lesssim \sum_{j\le N+3} (1+|\tau|)^{N+3-j}\|\nabla^j g_{high}\|_{\mathcal{LE}^*}\\ &\lesssim \sum_{j\le N+3} |\tau|^{N+3-j}\|\nabla^j g_{high}\|_{\mathcal{LE}^*}\\ &\lesssim \|\nabla^{N+3} g\|_{\mathcal{LE}^*}\end{aligned} \tag{2.21}$$

so (2.20) holds. Details of the calculations establishing the inequality

$$|\tau|^{N+3-j}\|\nabla^j g_{high}\|_{\mathcal{LE}^*} \lesssim \|\nabla^{N+3} g\|_{\mathcal{LE}^*} \tag{2.22}$$

used in (2.21) can be found in the Appendix (see section A.2.1). Because we are in the high frequency case, we have $|\tau| \le |\xi|$. Roughly speaking, on the Fourier transform side, we can turn powers of τ into powers of ξ which translate into derivatives when we invert the Fourier transform.

Continuing with the case $1 \lesssim |\tau|$, we now consider the low frequency part. We will use an iterative argument to decompose v_{low} into 2 pieces and establish the desired bounds for each piece separately.

Let Q_j denote a j-th order spatial operator with bounded coefficients (we allow Q_j to change from line to line, but these properties hold). Using this notation we can write

$$P_\tau = \tau^2 + \tau Q_1 + Q_2$$

so in general we have

$$P_\tau(R_\tau g - \tau^{-2} g) = \tau^{-1} Q_1 g + \tau^{-2} Q_2 g.$$

Thus we can estimate

$$v_{low} = \tau^{-2} g_{low} + v_1, \qquad P_\tau v_1 = \tau^{-1} Q_1 g_{low} + \tau^{-2} Q_2 g_{low} =: g_1.$$

Repeating the argument we find

$$v_1 = \tau^{-2} g_1 + v_2, \qquad P_\tau v_2 = \tau^{-1} Q_1 g_1 + \tau^{-2} Q_2 g_1 =: g_2.$$

Further iteration of the the argument yields

$$v_{low} = \sum_{j=0}^{M-1} \tau^{-2} g_j + v_M \tag{2.23}$$

where $g_0 = g_{low}$, $g_j = \tau^{-1} Q_1 g_{j-1} + \tau^{-2} Q_2 g_{j-2}$ and $v_M = R_\tau g_M$. Using our expression for g_j, we find

$$g_j = \sum_{\ell=j}^{2j} \tau^{-\ell} Q_\ell g_{low}$$

and plugging this into (2.23) yields

$$v_{low} = \sum_{j=0}^{2M-2} \tau^{-2-j} Q_j g_{low} + R_\tau \Big(\sum_{\ell=M}^{2M} \tau^{-\ell} Q_\ell g_{low}\Big).$$

We define

$$v_{low}^1 := \sum_{j=0}^{2M-2} \tau^{-2-j} Q_j g_{low}, \quad \text{and} \quad v_{low}^2 := R_\tau \Big(\sum_{\ell=M}^{2M} \tau^{-\ell} Q_\ell g_{low}\Big).$$

Our calculations establishing (2.19) will use the estimate

$$|\tau|^{-j} \|\nabla^{j+k} g_{low}\|_{\mathcal{LE}^*} = \sum_m 2^{\frac{m}{2}} |\tau|^{-j} \|\nabla^{j+k} g_{low}\|_{L^2(A_m)} \lesssim \|\nabla^k g\|_{\mathcal{LE}^*} \tag{2.24}$$

for $j \geq 0$. Details establishing this estimate can be found in the Appendix (see section A.2.2). Roughly speaking, since we are in the low frequency case, we have $|\tau|^{-1} \lesssim |\xi|^{-1}$ so powers of $|\tau|^{-1}$ can be used to remove derivatives.

To handle the v^1_{low} piece we calculate

$$\begin{aligned}
\|(\langle r\rangle^{-1}+|\tau|)v^1_{low}\|_{\mathcal{LE}^N} &\lesssim \sum_{j=0}^{2M-2} \|(\langle r\rangle^{-1}+|\tau|)\tau^{-2-j}Q_j g_{low}\|_{\mathcal{LE}^{*,N}} \\
&\lesssim \sum_{j=0}^{2M-2} \sum_{k\le N}\sum_m 2^{\frac{m}{2}}2^{-m}\||\tau|^{-j}\nabla^{j+k}g_{low}\|_{L^2(A_m)} \\
&\lesssim \sum_{j=0}^{2M-2} \sum_{k\le N} \|\nabla^k g\|_{\mathcal{LE}^*,} \\
&\lesssim \|g\|_{\mathcal{LE}^{*,N}}.
\end{aligned}$$

Similarly we find

$$\begin{aligned}
\|\nabla v^1_{low}\|_{\mathcal{LE}^N} &\lesssim \sum_{j=0}^{2M-2} \|\nabla\tau^{-2-j}Q_j g_{low}\|_{\mathcal{LE}^N} \\
&\lesssim \sum_{j=0}^{2M-2} \||\tau|^{-j-1}\nabla^{j+1}g_{low}\|_{\mathcal{LE}^N} \\
&\lesssim \|g\|_{\mathcal{LE}^{*,N}}.
\end{aligned}$$

Thus we have

$$\|(\langle r\rangle^{-1}+|\tau|)v^1_{low}\|_{\mathcal{LE}^N} + \|\nabla v^1_{low}\|_{\mathcal{LE}^N} \lesssim \|g\|_{\mathcal{LE}^{*,N}}. \tag{2.25}$$

To handle the v^2_{low} piece we apply (2.12) and calculate for $M > N+3$

$$\begin{aligned}
\|(\langle r\rangle^{-1}+|\tau|)v^2_{low}\|_{\mathcal{LE}^N} + \|\nabla v^2_{low}\|_{\mathcal{LE}^N} &\lesssim \sum_{\ell=M}^{2M}\sum_{j\le N+3}(1+|\tau|)^{N+3-j}\|\nabla^j\tau^{-\ell}Q_\ell g_{low}\|_{\mathcal{LE}^*} \\
&\lesssim \sum_{\ell=M}^{2M}\sum_{j\le N+3}|\tau|^{-(j+\ell-N-3)}\|\nabla^{j+\ell}g_{low}\|_{\mathcal{LE}^*} \\
&\lesssim \|g\|_{\mathcal{LE}^{*,N+3}}.
\end{aligned}$$

This bound, combined with (2.25), yields (2.19) as desired. This concludes the proof of (2.9).

Now we assume that P^2 is as in (2.10) and prove (2.11). Recall that the $\mathcal{LE}^N_\tau$ norm is defined by

$$\|v\|_{\mathcal{LE}^N_\tau} = \|(|\tau|+\langle r\rangle^{-1})v\|_{\mathcal{LE}^N} + \|\nabla v\|_{\mathcal{LE}^N} + \|(|\tau|+\langle r\rangle^{-1})^{-1}\nabla^2 v\|_{\mathcal{LE}^N}.$$

The first two terms are bounded using (2.9) and it is left to show that with the additional assumptions on P^2 the third term is similarly bounded. For this we again consider $|\tau| \lesssim 1$ and $|\tau| \gtrsim 1$ separately.

If $|\tau| \gtrsim 1$ or if $|\tau| \lesssim 1$ and $\langle r\rangle \lesssim 1$, then $(\langle r\rangle^{-1} + |\tau|)^{-1} \lesssim 1$, and by (2.9) we have

$$\|(\langle r\rangle^{-1} + |\tau|)^{-1}\nabla^2 v\|_{\mathcal{LE}^N} \lesssim \|\nabla^2 v\|_{\mathcal{LE}^N} \lesssim \|\nabla v\|_{\mathcal{LE}^{N+1}} \lesssim \|g\|_{\mathcal{LE}^{*,N+4}}.$$

We are left to consider the case where $\langle r\rangle$ is large and $|\tau| \lesssim 1$. For this we use the formula for P_τ to write

$$(\Delta + P^2)v = -\tau^2 v - i\tau P^1 v + g$$

and calculate

$$\|(\langle r\rangle^{-1} + |\tau|)^{-1}(\Delta + P^2)v\|_{\mathcal{LE}^N} \lesssim \||\tau|v\|_{\mathcal{LE}^N} + \|P^1 v\|_{\mathcal{LE}^N} + \|\langle r\rangle g\|_{\mathcal{LE}^N}.$$

The first two terms satisfy the desired bounds by (2.9). For the third term, straightforward calculation yields $\|\langle r\rangle g\|_{\mathcal{LE}^N} \lesssim \|g\|_{\mathcal{LE}^{*,N}}$ so we have

$$\|(\langle r\rangle^{-1} + |\tau|)^{-1}(\Delta + P^2)v\|_{\mathcal{LE}^N} \lesssim \|g\|_{\mathcal{LE}^{*,N+3}}.$$

Since $\kappa > 0$, the coefficients of P^2 tend to 0 as $r \to \infty$ so the principal part of $\Delta + P^2$ is elliptic for large r, and we can transfer the bounds on $(\Delta + P^2)v$ to $\nabla^2 v$ once we handle the lower order terms in P^2. The lower order terms in P^2 are bounded using (2.9). We show this explicitly for the terms

$$(\partial_i p_2^{ij})\partial_j, \quad \text{and} \quad V.$$

Using (2.9) we find

$$\|(\langle r\rangle^{-1} + |\tau|)^{-1}(\partial_i p_2^{ij})\partial_j v\|_{\mathcal{LE}^N} \lesssim \|\nabla v\|_{\mathcal{LE}^N} \lesssim \|g\|_{\mathcal{LE}^{*,N+3}}$$

since $p_2^{ij} \in \ell^1 S(1)$ and

$$\|(\langle r\rangle^{-1} + |\tau|)^{-1} V v\|_{\mathcal{LE}^N} \lesssim \|\langle r\rangle^{-1} v\|_{\mathcal{LE}^N} \lesssim \|g\|_{\mathcal{LE}^{*,N+3}}$$

since $V \in S(r^{-2})$. Thus we obtain

$$\|(\langle r\rangle^{-1} + |\tau|)^{-1}\nabla^2 v\|_{\mathcal{LE}^N} \lesssim \|g\|_{\mathcal{LE}^{*,N+3}}$$

as desired. This concludes the proof of (2.11). $\square$

2.3 Extending the Resolvent to $\tau \in \mathbb{R}$

Here we provide further assumptions that allow us to extend the resolvent $R_\tau^P = P_\tau^{-1}$ to $\tau \in \mathbb{R}$. We will use vector field methods to establish stronger resolvent bounds than those obtained in Proposition 2.4.

Proposition 2.5. *Let P be as in* (2.1) *with*

$$P^1 = \partial_i p_1^i(x) + p_1^i(x)\partial_i, \qquad P^2 = \partial_i p_2^{ij}(x)\partial_j + p_2^\omega(r)\Delta_\omega + V_\ell + V_r$$
$$p_1^i, p_2^{ij} \in \ell^1 S(r^{-\kappa}); \quad V_\ell \in \ell^1 S(r^{-\kappa-2}); \quad p_2^\omega, V_r \in S_{rad}(r^{-\kappa-2}).$$

Assume the Cauchy problem (2.2) *satisfies the uniform energy bounds* (1.4) *and the local energy estimate* (1.12)*.*

If $\kappa \geq 1$, $\Im\tau < 0$, and $g \in \mathcal{LE}^*$ satisfies

$$\|T^i\Omega^j S^k g\|_{\mathcal{LE}^{*,}} \lesssim 1, \quad i + 4j + 16k < M \tag{2.26}$$

for some positive integer M, then

$$\|T^i\Omega^j S^k (R_\tau g)\|_{\mathcal{LE}_\tau} \lesssim 1, \quad i + 4j + 16k < M - 4. \tag{2.27}$$

Proof. Take $v = R_\tau g$. To prove the proposition we will write

$$P_\tau T^i\Omega^j S^k v = T^i\Omega^j S^k g + [P_\tau, T^i\Omega^j S^k]v$$

and apply Proposition 2.4 to find

$$\|T^i\Omega^j S^k v\|_{\mathcal{LE}_\tau} \lesssim \|T^i\Omega^j S^k g\|_{\mathcal{LE}^{*,4}} + \|[P_\tau, T^i\Omega^j S^k]v\|_{\mathcal{LE}^{*,4}}.$$

Thus we need to calculate the commutator $[P_\tau, T^i\Omega^j S^k]$.

We use the notation Q_ℓ to denote any operator of the form

$$\tau(\partial_i h^i + h^i\partial_i) + \partial_i h^{ij}\partial_j + h_\ell, \qquad h^i, h^{ij} \in \ell^1 S(r^{-\kappa}), \quad h_\ell \in \ell^1 S(r^{-\kappa-2}) \tag{2.28}$$

and use Q_r to denote an operator of the form

$$h^\omega \Delta_\omega + h_r, \quad h^\omega, h_r \in S_{rad}(r^{-\kappa-2}). \tag{2.29}$$

The use of the subscript ℓ indicates the coefficients are ℓ^1 summable, and the subscript r indicates the coefficients are radial functions. We allow the coefficients of the operators Q_ℓ and Q_r to change, but the properties described above must still hold. Note our assumptions on P mean P_τ can be written in this notation as

$$P_\tau = \tau^2 + \Delta + Q_\ell + Q_r.$$

Consider the commutator $[P_\tau, \Omega]$. For $\rho^q_\ell \in \ell^1 S(r^q)$ and $\rho^q_r \in S_{rad}(r^q)$ we find

$$[\tau^2, \Omega] = [\Delta, \Omega] = [\rho^q_r, \Omega] = [\rho^q_r \Delta_\omega, \Omega] = 0,$$
$$[\rho^q_\ell, \Omega] = \rho^q_\ell, \quad [\rho^q_\ell \partial_x, \Omega] = \rho^q_\ell \partial_x, \quad \text{and} \quad [\rho^q_\ell \partial^2_x, \Omega] = \rho^q_\ell \partial^2_x$$

where we use ρ^q_ℓ and ρ^q_r as representatives of their respective symbol classes that may indicate different functions each time they appear. We will be concerned only with the space in which $[P_\tau, T^i\Omega^j S^k]v$ lies so only the symbol classes (not the precise functions) are relevant to our calculations. It follows that

$$[P_\tau, \Omega] = Q_\ell \quad \text{and} \quad [Q_\ell, \Omega] = Q_\ell.$$

Similarly we find

$$[\Delta, S] = 2\Delta, \quad [\tau^2, S] = 2\tau^2,$$
$$[i\tau P^1, S] = 2i\tau P^1 + Q_\ell, \quad \text{and} \quad [P^2, S] = 2P^2 + Q_\ell + Q_r.$$

It follows that

$$[P_\tau, S] = 2P_\tau + Q_\ell + Q_r, \quad [Q_r, S] = Q_r, \quad \text{and} \quad [Q_\ell, S] = Q_\ell.$$

Finally, we find

$$[\Delta, T] = [\tau^2, T] = 0$$

$$[Q_\ell, T] = Q_\ell, \quad [Q_r, T] = Q_r, \quad \text{and} \quad [P_\tau, T] = Q_\ell + Q_r.$$

Now we are ready to calculate $[P_\tau, T^i\Omega^j S^k]$. We find

$$\begin{aligned}
[P_\tau, T^i\Omega^j S^k] &= \sum_{n=0}^{i-1} T^n(Q_\ell + Q_r)T^{i-n-1}\Omega^j S^k + \sum_{n=0}^{j-1} T^i\Omega^n(Q_\ell)T^{j-n-1}S^k \\
&\quad + \sum_{n=0}^{k-1} T^i\Omega^j S^n(2P_\tau + Q_\ell + Q_r)S^{k-n-1} \\
&= (Q_\ell + Q_r)\left(\sum_{n=0}^{i-1} T^n\right)\Omega^j S^k + Q_\ell\left(\sum_{n=0}^{i} T^n\right)\left(\sum_{n=0}^{j-1}\Omega^n\right)S^k + c_n T^i\Omega^j\left(\sum_{n=0}^{k-1} S^n\right)P_\tau \\
&\quad + Q_\ell\left(\sum_{n=0}^{i} T^n\right)\left(\sum_{n=0}^{j}\Omega^n\right)\left(\sum_{n=0}^{k-1} S^n\right) + Q_r\left(\sum_{p=0}^{i} T^p\right)\Omega^j\left(\sum_{n=0}^{k-1} S^n\right) \\
&= T^i\Omega^j S^{<k}P_\tau + Q_\ell\left(T^{<i}\Omega^j S^k + T^{\le i}\Omega^{<j}S^k + T^{\le i}\Omega^{\le j}S^{<k}\right) \\
&\quad + Q_r\left(T^{<i}\Omega^j S^k + T^{\le i}\Omega^j S^{<k}\right).
\end{aligned}$$

In the last line we use the notation $\Gamma^{<i} = \sum_{|\alpha|=0}^{i-1} c_\alpha\Gamma^\alpha$. Note we define $\Gamma^{<i}$ to be a linear combination of Γ^α for $|\alpha| < i$ to account for the constant which arises when commuting P_τ with S. Take $v_{ijk} = T^i\Omega^j S^k v$ and $g_{ijk} = T^i\Omega^j S^k g$. Similarly, we write $v_{<i<j<k} = T^{<i}\Omega^{<i}S^{<k}v$ and use analogous notation for g. Using this notation and the commutator calculated above, we have

$$P_\tau v_{ijk} = g_{ij\le k} + Q_\ell(v_{<ijk} + v_{\le i<jk} + v_{\le i\le j<k}) + Q_r(v_{<ijk} + v_{\le ij<k}). \tag{2.30}$$

By Proposition 2.4 we have that $\|v_{ijk}\|_{\mathcal{LE}_\tau}$ is controlled by the right hand side of (2.30) measured in the $\mathcal{LE}^{*,4}$ norm. To obtain useful bounds on the terms on the right hand side of (2.30) with a v component, we relate the $\mathcal{LE}^*$ and $\mathcal{LE}$ norms. If $\rho_\ell^q \in \ell^1S(r^q)$ and $\rho_r^q \in S_{rad}(r^q)$, then by Lemma A.1 we have

$$\|\rho_\ell^q v\|_{\mathcal{LE}^*} \lesssim \|v\|_{\mathcal{LE}}, \quad q \le -1$$

and

$$\|\rho_r^q v\|_{\mathcal{LE}^*} \lesssim \|v\|_{\mathcal{LE}}, \quad q < -1.$$

It follows that

$$\begin{aligned}\|Q_\ell v\|_{\mathcal{LE}^*} &\lesssim \||\tau|\nabla v\|_{\mathcal{LE}} + \|\tau\langle r\rangle^{-1}v\|_{\mathcal{LE}} + \|\nabla^2 v\|_{\mathcal{LE}} + \|\langle r\rangle^{-1}\nabla v\|_{\mathcal{LE}} + \|\langle r\rangle^{-2}v\|_{\mathcal{LE}} \\ &\lesssim \|(\langle r\rangle^{-1} + |\tau|)v\|_{\mathcal{LE}^1} + \|\nabla v\|_{\mathcal{LE}^1}\end{aligned}$$

and

$$\|Q_r v\|_{\mathcal{LE}^*} \lesssim \|\langle r\rangle^{-1}v\|_{\mathcal{LE}} + \|\langle r\rangle^{-1}\Omega^2 v\|_{\mathcal{LE}}$$

where we have used the fact $|\Delta_\omega v| \lesssim |\Omega^2 v|$. Thus we see

$$\begin{aligned}&\|Q_\ell v\|_{\mathcal{LE}^{*,N}} \lesssim \|(\langle r\rangle^{-1} + |\tau|)v\|_{\mathcal{LE}^{N+1}} + \|\nabla v\|_{\mathcal{LE}^{N+1}} =: \|v\|_{\mathcal{LE}^{N+1}_{\tau,1}} \\ &\|Q_r v\|_{\mathcal{LE}^{*,N}} \lesssim \|v\|_{\mathcal{LE}^N_\tau} + \|\Omega^2 v\|_{\mathcal{LE}^N_\tau}.\end{aligned} \tag{2.31}$$

Here we have defined $\mathcal{LE}^N_{\tau,1}$ to be the first two terms in the $\mathcal{LE}_\tau$ norm, which are bounded using (2.9).

Note by Proposition 2.4 we have

$$\|v\|_{\mathcal{LE}^{N+1}_{\tau,1}} + \|v\|_{\mathcal{LE}^N_\tau} \lesssim \|g\|_{\mathcal{LE}^{*,N+4}}.$$

Therefore we see

$$\|v_{i00}\|_{\mathcal{LE}_\tau} \le \|v\|_{\mathcal{LE}^i_\tau} \lesssim \|g\|_{\mathcal{LE}^{*,i+4}} \lesssim 1, \qquad i < M-4. \tag{2.32}$$

The rest of the proof proceeds as follows. We will use the v_{i00} bound (2.32) to show $\|v_{ij0}\|_{\mathcal{LE}_\tau} \lesssim 1$ for $i + 4j < M - 4$. Then we will prove $\|v_{ijk}\|_{\mathcal{LE}_\tau} \lesssim 1$ for $i + 4j + 16k < M - 4$, as stated in the proposition. We will use the bound

$$\|v_{ijk}\|_{\mathcal{LE}_\tau} \lesssim \|v_{0jk}\|_{\mathcal{LE}^i_\tau}$$

and proceed using v_{0jk}. Note that for v_{0jk}, (2.30) reduces to

$$P_\tau v_{0jk} = g_{0j\le k} + Q_\ell(v_{0<jk} + v_{0\le j<k}) + Q_r v_{0j<k}. \tag{2.33}$$

Consider $\|v_{ij0}\|_{\mathcal{LE}_\tau}$ for $i+4j < M-4$. We need only consider $j \geq 1$ since we have already shown $\|v_{i00}\|_{\mathcal{LE}_\tau} \lesssim 1$. By (2.33) we see

$$P_\tau v_{0j0} = g_{0j0} + Q_\ell v_{0<j0}$$

and we find using (2.11) and (2.31)

$$\begin{aligned}
\|v_{ij0}\|_{\mathcal{LE}_\tau} &\lesssim \|v_{0j0}\|_{\mathcal{LE}^i_\tau} \\
&\lesssim \|g_{0j0}\|_{\mathcal{LE}^{*,i+4}} + \|Q_\ell v_{0<j0}\|_{\mathcal{LE}^{*,i+4}} \\
&\lesssim \|g_{0j0}\|_{\mathcal{LE}^{*,i+4}} + \|v_{0<j0}\|_{\mathcal{LE}^{i+5}_{\tau,1}}.
\end{aligned}$$

To handle the last term on the right hand side we will use induction to prove for all $j \geq 1$

$$\|v_{0j0}\|_{\mathcal{LE}^i_{\tau,1}} \lesssim \sum_{a=0}^{j} \|g_{0a0}\|_{\mathcal{LE}^{*,i+3+4(j-a)}}. \tag{2.34}$$

Once (2.34) is established, we have

$$\|v_{ij0}\|_{\mathcal{LE}_\tau} \lesssim \|g_{0j0}\|_{\mathcal{LE}^{*,i+4}} + \sum_{a=0}^{j-1} \|g_{0a0}\|_{\mathcal{LE}^{*,i+4+4(j-a)}}.$$

The restriction $i+4j < M-4$ gives $i+4+4j < M$ so the right hand side is bounded by assumption.

Thus we need to prove (2.34) to establish $\|v_{ij0}\|_{\mathcal{LE}_\tau} \lesssim 1$. If $j=1$, then we have

$$P_\tau v_{010} = g_{010} + Q_\ell v_{000}$$

and by (2.9) and (2.31) we have

$$\begin{aligned}
\|v_{010}\|_{\mathcal{LE}^i_{\tau,1}} &\lesssim \|g_{010}\|_{\mathcal{LE}^{*,i+3}} + \|Q_\ell v\|_{\mathcal{LE}^{*,i+3}} \\
&\lesssim \|g_{010}\|_{\mathcal{LE}^{*,i+3}} + \|v\|_{\mathcal{LE}^{i+4}_{\tau,1}} \\
&\lesssim \|g_{010}\|_{\mathcal{LE}^{*,i+3}} + \|g\|_{\mathcal{LE}^{*,i+7}} \\
&= \sum_{a=0}^{1} \|g_{0a0}\|_{\mathcal{LE}^{*,i+3+4(1-a)}}
\end{aligned}$$

as desired. Now fix J and assume (2.34) holds for all $j < J$. By (2.33) we have

$$P_\tau v_{0J0} = g_{0J0} + Q_\ell v_{0<J0}.$$

Using (2.9), (2.31), and the inductive hypothesis we obtain

$$\begin{aligned}
\|v_{0J0}\|_{\mathcal{LE}^i_{\tau,1}} &\lesssim \|g_{0J0}\|_{\mathcal{LE}^{*,i+3}} + \|Q_\ell v_{0<J0}\|_{\mathcal{LE}^{*,i+3}} \\
&\lesssim \|g_{0J0}\|_{\mathcal{LE}^{*,i+3}} + \|v_{0<J0}\|_{\mathcal{LE}^{i+4}_{\tau,1}} \\
&\lesssim \|g_{0J0}\|_{\mathcal{LE}^{*,i+3}} + \sum_{a=0}^{J-1} \|g_{0a0}\|_{\mathcal{LE}^{*,i+3+4(J-a)}} \\
&= \sum_{a=0}^{J} \|g_{0a0}\|_{\mathcal{LE}^{*,i+3+4(J-a)}}.
\end{aligned}$$

This concludes the proof of (2.34) and thus establishes $\|v_{ij0}\|_{\mathcal{LE}_\tau} \lesssim 1$ for $i + 4j < M - 4$.

Finally we consider $\|v_{ijk}\|_{\mathcal{LE}_\tau}$ for $i + 4j + 16k < M - 4$. When $k = 0$ we have already shown

$$\|v_{ij0}\|_{\mathcal{LE}_\tau} \lesssim 1, \quad i + 4j + 16(0) = i + 4j < M - 4.$$

Fix K such that $i + 4j + 16K < M - 4$ and assume for $k < K$ we have

$$\|v_{ijk}\|_{\mathcal{LE}_\tau} \lesssim 1, \qquad i + 4j + 16k < M - 4.$$

By (2.33), we have

$$P_\tau v_{0jK} = g_{0j\le K} + Q_\ell(v_{0<jK} + v_{0\le j<K}) + Q_r v_{0j<K}$$

so that

$$\begin{aligned}
\|v_{ijK}\|_{\mathcal{LE}_\tau} &\lesssim \|v_{0jK}\|_{\mathcal{LE}^i_\tau} \\
&\lesssim \|g_{0j\le K}\|_{\mathcal{LE}^{*,i+4}} + \|Q_\ell v_{0<jK}\|_{\mathcal{LE}^{*,i+4}} + \|Q_\ell v_{0\le j<K}\|_{\mathcal{LE}^{*,i+4}} + \|Q_r v_{0j<K}\|_{\mathcal{LE}^{*,i+4}} \\
&\lesssim \|g_{0j\le K}\|_{\mathcal{LE}^{*,i+4}} + \|v_{0<jK}\|_{\mathcal{LE}^{i+5}_{\tau,1}} + \|v_{0\le j<K}\|_{\mathcal{LE}^{i+5}_{\tau,1}} + \|v_{0j<K}\|_{\mathcal{LE}^{i+4}_\tau} \\
&\quad + \|v_{0(j+2)<K}\|_{\mathcal{LE}^{i+4}_\tau}.
\end{aligned}$$

If $i+4j+16K < M-4$, then the first term is bounded by assumption and the last three terms are bounded by the inductive hypothesis. It is left to consider $\|v_{0<jK}\|_{\mathcal{LE}^{i+5}_{\tau,1}}$. This term arises only when $j \geq 1$.

If $j=1$ then $v_{0<jK} = v_{00K}$ and our assumption that $i+4j+16K < M-4$ gives $i+16K < M-8$.

By (2.33) we have

$$P_\tau v_{00K} = g_{00\leq K} + Q_\ell v_{00<K} + Q_r v_{00<K}$$

so that

$$\begin{aligned}\|v_{00K}\|_{\mathcal{LE}^{i+5}_{\tau,1}} &\lesssim \|g_{00\leq K}\|_{\mathcal{LE}^{*,i+8}} + \|Q_\ell v_{00<K}\|_{\mathcal{LE}^{*,i+8}} + \|Q_r v_{00<K}\|_{\mathcal{LE}^{*,i+8}} \\ &\lesssim \|g_{00\leq K}\|_{\mathcal{LE}^{*,i+8}} + \|v_{00<K}\|_{\mathcal{LE}^{i+9}_{\tau,1}} + \|v_{00<K}\|_{\mathcal{LE}^{i+8}_{\tau}} + \|v_{02<K}\|_{\mathcal{LE}^{i+8}_{\tau}}.\end{aligned}$$

Since $i+16K < M-8$ the first term is bounded by assumption, and the last three terms are bounded by the inductive hypothesis on $k < K$.

Now fix J and assume that if $j < J$ then $\|v_{0jK}\|_{\mathcal{LE}^{i+5}_{\tau,1}} \lesssim 1$ for $i+4J+16K < M-8$. We find

$$\begin{aligned}\|v_{0JK}\|_{\mathcal{LE}^{i+5}_{\tau,1}} &\lesssim \|g_{0J\leq K}\|_{\mathcal{LE}^{*,i+8}} + \|Q_\ell v_{0<JK}\|_{\mathcal{LE}^{*,i+8}} + \|Q_\ell v_{0\leq J<K}\|_{\mathcal{LE}^{*,i+8}} + \|Q_r v_{0J<K}\|_{\mathcal{LE}^{*,i+8}} \\ &\lesssim \|g_{0J\leq K}\|_{\mathcal{LE}^{*,i+8}} + \|v_{0<JK}\|_{\mathcal{LE}^{i+9}_{\tau,1}} + \|v_{0\leq J<K}\|_{\mathcal{LE}^{i+9}_{\tau,1}} + \|v_{0J<K}\|_{\mathcal{LE}^{i+8}_{\tau}} \\ &\quad + \|v_{0(J+2)<K}\|_{\mathcal{LE}^{i+8}_{\tau}}.\end{aligned}$$

For $i+4J+16K < M-8$, the first term is bounded by assumption. The second term is bounded by the inductive hypothesis on $j < J$. Since $\|v_{0\leq J<K}\|_{\mathcal{LE}^{i+9}_{\tau,1}} \lesssim \|v_{0\leq J<K}\|_{\mathcal{LE}^{i+9}_{\tau}}$, the third term is bounded by the inductive hypothesis on $k < K$. The last two terms are similarly bounded by the inductive hypothesis on $k < K$.

Therefore $\|v_{0<jK}\|_{\mathcal{LE}^{i+5}_{\tau,1}} \lesssim 1$ if $i+4(j-1)+16K < M-8$, which concludes the proof that $\|v_{ijk}\|_{\mathcal{LE}_\tau} \lesssim 1$. When $i+4j+16k < M-4$. $\square$

For the remaining propositions, let P be as in (2.1) with

$$\begin{aligned}P^1 = \partial_i p_1^i + p_1^i \partial_i, \qquad P^2 = \partial_i p_2^{ij}(x)\partial_j + p_2^\omega(r)\Delta_\omega + V_\ell + V_r \\ p_1^i, p_2^{ij} \in \ell^1 S(r^{-1}); \quad V_\ell \in \ell^1 S(r^{-3}); \quad p_2^\omega, V_r \in S_{rad}(r^{-3})\end{aligned}$$

and assume the Cauchy problem (2.2) satisfies the uniform energy bounds (1.4) and the local energy estimate (1.12).

Proposition 2.6 (see [33, Proposition 11]). *1. The operators R_τ extend continuously to $\tau \in \mathbb{R} \setminus \{0\}$ in the $H^4_{comp} \to L^2_{loc}$ topology. Furthermore the inequality* (2.11) *holds for $\tau \in \mathbb{R} \setminus \{0\}$.*

2. If $\tau \in \mathbb{R} \setminus \{0\}$ and $g \in \mathcal{LE}^*$, then for $v = R_\tau g$ the radiation condition

$$\lim_{m\to\infty} 2^{-\frac{m}{2}} \|(\partial_r + i\tau)v\|_{L^2(A_m)} = 0 \tag{2.35}$$

holds.

3. Let $\tau \in \mathbb{R} \setminus \{0\}$ and suppose $v \in \mathcal{LE}^4_\tau$ satisfies the outgoing radiation condition (2.35). *If $P_\tau v = g \in \mathcal{LE}^{*,4}$ then $v = R_\tau g$.*

From these results we obtain the following corollary:

Corollary 2.7 (see [33, Corollary 12]). *Let P be as above and let $\tau \in \mathbb{R} \setminus \{0\}$. The bounds in Proposition 2.4 and Proposition 2.5 hold. Furthermore, for i, j, k as in Proposition 2.5, the functions $T^i \Omega^j S^k R_\tau g$ satisfy the outgoing radiation condition* (2.35).

Proposition 2.8 (see [33, Proposition 13]). *1. Let $g, \langle r \rangle g \in \mathcal{LE}^{*,4}$. Then the limit*

$$R_0 g = \lim_{\epsilon \to 0} R_{-i\epsilon} g$$

exists in the L^2_{loc} topology, and the following bound holds:

$$\|\langle r \rangle R_0 g\|_{\mathcal{LE}^n_0} \lesssim \|\langle r \rangle g\|_{\mathcal{LE}^{*,n+4}}, \quad n \geq 0.$$

2. The operator R_0 admits a unique continuous extension to $\mathcal{LE}^{*,4}$ which satisfies

$$\|R_0 g\|_{\mathcal{LE}^n_0} \lesssim \|g\|_{\mathcal{LE}^{*n+4}}, \quad n \geq 0$$

and

$$\lim_{m\to\infty} 2^{(j-\frac{3}{2}m)} \|\nabla^j R_0 g\|_{L^2(A_m)} = 0, \quad m = 0, 1, 2.$$

3. Let $v \in \mathcal{LE}_0^4$ *so that* $P_0 v = g \in \mathcal{LE}^{*,4}$ *and*

$$\lim_{m\to\infty} 2^{-\frac{m}{2}} \|\partial_r v\|_{L^2(A_m)} = 0.$$

Then $v = R_0 g$.

CHAPTER 3

Coordinate change

In this chapter we will show there exists a normalized coordinate system in which the operator $\Box_g + V$ used in the statement of the main theorem can be replaced by an operator P of the form (2.1). The coefficients of the operators P^1 and P^2 will depend on the metric coefficients. We will see that P satisfies the assumptions of the propositions in Chapter 2. The calculations in this chapter encode the geometric assumptions into the differential operator, and we work in these coordinates throughout the rest of the paper. The statement of the main theorem is given in these normalized coordinates.

For reference, we recall the metric assumptions laid out in Chapter 1. Let $\mathfrak{g}$ be a non-degenerate Lorentzian metric with signature $(3, 1)$. Furthermore, assume $\mathfrak{g}$ satisfies the following:

1. The metric $\mathfrak{g}$ is stationary, meaning the metric coefficients are time independent.

2. The submanifolds $t = constant$ are space-like (i.e. the metric on the spatial submanifolds is positive definite).

3. Let $\kappa \in \mathbb{N}$ with $\kappa \geq 2$. The metric $\mathfrak{g}$ is asymptotically flat in the sense that $\mathfrak{g}$ can be written as

$$\mathfrak{g} = \mathfrak{m} + \mathfrak{f} + \mathfrak{h},$$

where $\mathfrak{m}$ is the flat Minkowski metric

$$\mathfrak{m} = -dt^2 + \sum_{i=1}^{3} dx_i^2,$$

$\mathfrak{f}$ is given by

$$\mathfrak{f} = \mathfrak{f}_{00}(x)dt^2 + \mathfrak{f}_{0i}(x)dtdx_i + \mathfrak{f}_{ij}(x)dx_idx_j$$

with $\mathfrak{f}_{\alpha\beta} \in \ell^1 S(r^{-\kappa})$ for $\alpha, \beta \in \{0, 1, 2, 3\}$, and $\mathfrak{h}$ is given by

$$\mathfrak{h} = \mathfrak{h}_{tt}(r)dt^2 + \mathfrak{h}_{tr}(r)dtdr + \mathfrak{h}_{rr}(r)dr^2 + \mathfrak{h}_{\omega\omega}(r)r^2 d\omega^2$$

with $\mathfrak{h}_{\gamma\delta} \in S_{rad}(r^{-\kappa})$ for $\gamma, \delta \in \{t, r, \omega\}$. Here $d\omega^2 = d\theta^2 + \sin^2\theta d\phi^2$.

In the assumptions above and throughout this chapter we use α, β when working in standard coordinates and γ, δ when working in spherical coordinates.

Assumption 3 can be restated in spherical coordinates as

$$\mathfrak{g} = \mathfrak{m} + \mathfrak{f} + \mathfrak{h},$$

where the flat Minkowski metric $\mathfrak{m}$ is given by

$$\mathfrak{m} = -dt^2 + dr^2 + r^2 d\omega^2,$$

$\mathfrak{f}$ is given by

$$\begin{aligned}\mathfrak{f} = {} & \mathfrak{f}_{tt}dt^2 + \mathfrak{f}_{tr}dtdr + \mathfrak{f}_{t\theta}rdtd\theta + \mathfrak{f}_{t\phi}r\sin\theta dtd\phi + \mathfrak{f}_{rr}dr^2 + \mathfrak{f}_{r\theta}rdrd\theta + \mathfrak{f}_{r\phi}r\sin\theta drd\phi \\ & + \mathfrak{f}_{\theta\theta}r^2 d\theta^2 + \mathfrak{f}_{\theta\phi}r^2\sin\theta d\theta d\phi + \mathfrak{f}_{\phi\phi}r^2\sin^2\theta d\phi d\phi\end{aligned}$$

with $\mathfrak{f}_{\gamma\delta} \in \ell^1 S(r^{-\kappa})$ for $\gamma, \delta \in \{t, r, \theta, \phi\}$ and

$$\mathfrak{h} = \mathfrak{h}_{tt}(r)dt^2 + \mathfrak{h}_{tr}(r)dtdr + \mathfrak{h}_{rr}(r)dr^2 + \mathfrak{h}_{\omega\omega}(r)r^2 d\omega^2$$

with $\mathfrak{h}_{\gamma\delta} \in S_{rad}(r^{-\kappa})$ for $\gamma, \delta \in \{t, r, \omega\}$.

Alternatively, Assumption 3 can be restated in dual coefficients (still in spherical coordinates) as

$$\mathfrak{g}^{\gamma\delta} = \mathfrak{m}^{\gamma\delta} + \mathfrak{f}^{\gamma\delta} + \mathfrak{h}^{\gamma\delta}$$

where

$$\left[\mathfrak{m}^{\gamma\delta}\right] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{bmatrix}, \qquad \left[\mathfrak{f}^{\gamma\delta}\right] = \begin{bmatrix} \mathfrak{f}^{tt} & \mathfrak{f}^{tr} & \frac{f^{t\theta}}{r} & \frac{f^{t\phi}}{r\sin\theta} \\ \mathfrak{f}^{rt} & \mathfrak{f}^{rr} & \frac{f^{r\theta}}{r} & \frac{f^{r\phi}}{r\sin\theta} \\ \frac{\mathfrak{f}^{\theta t}}{r} & \frac{\mathfrak{f}^{\theta r}}{r} & \frac{f^{\theta\theta}}{r^2} & \frac{f^{\theta\phi}}{r^2\sin\theta} \\ \frac{\mathfrak{f}^{\phi t}}{r\sin\theta} & \frac{\mathfrak{f}^{\phi r}}{r\sin\theta} & \frac{f^{\phi\theta}}{r^2\sin\theta} & \frac{f^{\phi\phi}}{r^2\sin^2\theta} \end{bmatrix},$$

and

$$\left[\mathfrak{h}^{\gamma\delta}\right] = \begin{bmatrix} \mathfrak{h}^{tt} & \mathfrak{h}^{tr} & 0 & 0 \\ \mathfrak{h}^{rt} & \mathfrak{h}^{rr} & 0 & 0 \\ 0 & 0 & \frac{\mathfrak{h}^{\omega\omega}}{r^2} & 0 \\ 0 & 0 & 0 & \frac{\mathfrak{h}^{\omega\omega}}{r^2\sin^2\theta} \end{bmatrix}$$

with $\mathfrak{f}^{\gamma\delta} \in \ell^1 S(r^{-\kappa})$ and $\mathfrak{h}^{\gamma\delta} \in S_{rad}(r^{-\kappa})$. The calculations establishing both restatements of Assumption 3 can be found in the Appendix (see section A.3).

Lemma 3.1. *There exists a coordinate system so that $\mathfrak{g}$ satisfies Assumptions 1-3 above as well as the additional condition*

$$\mathfrak{h}^{rr} = -\mathfrak{h}^{tt}, \quad \mathfrak{h}^{tr} = 0.$$

Proof. In order to achieve $\mathfrak{h}^{tr} = 0$, we reset t via the coordinate change

$$dT = dt - \chi_{>R}\frac{\mathfrak{h}^{rt}}{1+\mathfrak{h}^{rr}}dr$$

where $\chi_{>R} = \chi_{>1}\left(\frac{|x|}{R}\right)$ and R is a constant chosen so that $1 + \mathfrak{h}^{rr} \gtrsim 1$ for $r > R$. Define

$$q(r) := \chi_{>R}\frac{\mathfrak{h}^{rt}}{1+\mathfrak{h}^{rr}}.$$

Since $\mathfrak{h}^{rt} \in S_{rad}(r^{-\kappa})$ and since $\chi_{>R}\frac{1}{1+\mathfrak{h}^{rr}} \lesssim 1$ by our choice of the constant R, we see $q(r) \in S_{rad}(r^{-\kappa})$.

To see Assumption 1 holds under this coordinate change, write the coordinate change as

$$T = t + Q(r)$$

where $Q'(r) = -q(r)$. We calculate $\frac{\partial}{\partial T}$ and find

$$\frac{\partial}{\partial T} = \frac{\partial t}{\partial T}\frac{\partial}{\partial t} = \frac{\partial}{\partial t}.$$

Thus

$$\frac{\partial}{\partial T}\mathfrak{g}^{\gamma\delta} = \frac{\partial}{\partial t}\mathfrak{g}^{\gamma\delta} = 0$$

so the metric coefficients remain independent of the time variable T.

To see Assumption 2 still holds, we calculate

$$\langle dT, dT\rangle = \langle dt, dt\rangle - 2q(r)\langle dt, dr\rangle + (q(r))^2\langle dr, dr\rangle.$$

Choosing R sufficiently large so that q is sufficiently small, we the sign of $\mathfrak{g}^{TT}$ is the same as the sign of $\mathfrak{g}^{tt}$. The signature of the metric does not change under the change of coordinates, so the $t = \textit{constant}$ submanifolds remain positive definite.

We now calculate $\mathfrak{g}^{\gamma\delta}$ in the new coordinate system. Since r, θ, and ϕ are unchanged, we need only calculate $\mathfrak{g}^{T\gamma}$ for $\gamma \in \{T, r, \theta, \phi\}$. First we calculate $\mathfrak{g}^{TT}$ and find

$$\begin{aligned}\mathfrak{g}^{TT} &= \mathfrak{g}^{tt} - 2q(r)\mathfrak{g}^{tr} + (q(r))^2\mathfrak{g}^{rr} \\ &= -1 + \mathfrak{f}^{tt} + \mathfrak{h}^{tt} - 2q(r)(\mathfrak{f}^{tr} + \mathfrak{h}^{tr}) + (q(r))^2(1 + \mathfrak{f}^{rr} + \mathfrak{h}^{rr}).\end{aligned}$$

Taking

$$\tilde{\mathfrak{f}}^{TT} := \mathfrak{f}^{tt} - 2q(r)(\mathfrak{f}^{tr} + \mathfrak{h}^{tr}) + (q(r))^2(1 + \mathfrak{f}^{rr} + \mathfrak{h}^{rr})$$

and

$$\tilde{\mathfrak{h}}^{TT} := \mathfrak{h}^{tt}$$

we find

$$\mathfrak{g}^{TT} = -1 + \tilde{\mathfrak{f}}^{TT} + \tilde{\mathfrak{h}}^{TT} \tag{3.1}$$

where $\tilde{\mathfrak{f}}^{TT} \in \ell^1 S(r^{-\kappa})$ and $\tilde{\mathfrak{h}}^{TT} \in S_{rad}(r^{-\kappa})$.

Next we calculate $\mathfrak{g}^{Tr}$ and find

$$\begin{aligned}
\mathfrak{g}^{Tr} &= \mathfrak{g}^{tr} - q(r)\mathfrak{g}^{rr} \\
&= \mathfrak{f}^{tr} + \mathfrak{h}^{tr} - \chi_{>R}\frac{\mathfrak{h}^{tr}}{1+\mathfrak{h}^{rr}}(1 + \mathfrak{f}^{rr} + \mathfrak{h}^{rr}) \\
&= \mathfrak{f}^{tr} + \mathfrak{h}^{tr}(1 - \chi_{>R}) - \chi_{>R}\frac{\mathfrak{h}^{tr}}{1+\mathfrak{h}^{rr}}\mathfrak{f}^{rr} \\
&= \mathfrak{f}^{tr} - q(r)\mathfrak{f}^{rr} + \chi_{<R}\mathfrak{h}^{tr}.
\end{aligned}$$

Since $\mathfrak{f}^{tr}, \mathfrak{f}^{rr} \in \ell^1 S(r^{-\kappa})$, $q \in S_{rad}(r^{-\kappa})$, and $\chi_{<R}$ is compactly supported, we see in the new coordinates $\mathfrak{g}^{Tr} \in \ell^1 S(r^{-\kappa})$. Thus we can define

$$\tilde{\mathfrak{f}}^{Tr} := \mathfrak{f}^{tr} - q(r)\mathfrak{f}^{rr} + \chi_{<R}\mathfrak{h}^{tr}$$

and write

$$\mathfrak{g}^{Tr} = \tilde{\mathfrak{f}}^{Tr} \tag{3.2}$$

and retain the desired property that $\tilde{\mathfrak{f}}^{Tr} \in \ell^1 S(r^{-\kappa})$.

Next we calculate $\mathfrak{g}^{T\theta}$ and find

$$\begin{aligned}
\mathfrak{g}^{T\theta} &= \mathfrak{g}^{t\theta} - q(r)\mathfrak{g}^{r\theta} \\
&= \frac{\mathfrak{f}^{t\theta} - q(r)\mathfrak{f}^{r\theta}}{r}.
\end{aligned}$$

Note $\mathfrak{f}^{t\theta}, \mathfrak{f}^{r\theta} \in \ell^1 S(r^{-\kappa})$ and $q \in S_{rad}(r^{-\kappa})$. Thus we define

$$\tilde{\mathfrak{f}}^{T\theta} := \mathfrak{f}^{t\theta} - q(r)\mathfrak{f}^{r\theta}$$

so that

$$\mathfrak{g}^{T\theta} = \frac{\tilde{\mathfrak{f}}^{T\theta}}{r} \tag{3.3}$$

where $\tilde{\mathfrak{f}}^{T\theta} \in \ell^1 S(r^{-\kappa})$.

Finally we calculate $\mathfrak{g}^{T\phi}$ and find

$$\begin{aligned} \mathfrak{g}^{T\phi} &= \mathfrak{g}^{t\phi} - q(r)\mathfrak{g}^{r\phi} \\ &= \frac{\mathfrak{f}^{t\phi} - q(r)\mathfrak{f}^{r\phi}}{r\sin\theta}. \end{aligned}$$

Note $\mathfrak{f}^{t\phi}, \mathfrak{f}^{r\phi} \in \ell^1 S(r^{-\kappa})$ and $q \in S_{rad}(r^{-\kappa})$. Thus we define

$$\tilde{\mathfrak{f}}^{T\phi} := \mathfrak{f}^{t\phi} - q(r)\mathfrak{f}^{r\phi}$$

so that

$$\mathfrak{g}^{T\phi} = \frac{\tilde{\mathfrak{f}}^{T\phi}}{r\sin\theta} \tag{3.4}$$

where $\tilde{\mathfrak{f}}^{T\phi} \in \ell^1 S(r^{-\kappa})$.

Combining (3.1), (3.2), (3.3), and (3.4) and relabeling yields

$$\left[\mathfrak{g}^{\gamma\delta}\right] = \left[\mathfrak{m}^{\gamma\delta}\right] + \begin{bmatrix} \mathfrak{f}^{tt} & \mathfrak{f}^{tr} & \frac{f^{t\theta}}{r} & \frac{f^{t\phi}}{r\sin\theta} \\ \mathfrak{f}^{rt} & \mathfrak{f}^{rr} & \frac{f^{r\theta}}{r} & \frac{f^{r\phi}}{r\sin\theta} \\ \frac{\mathfrak{f}^{\theta t}}{r} & \frac{\mathfrak{f}^{\theta r}}{r} & \frac{f^{\theta\theta}}{r^2} & \frac{f^{\theta\phi}}{r^2\sin\theta} \\ \frac{\mathfrak{f}^{\phi t}}{r\sin\theta} & \frac{\mathfrak{f}^{\phi r}}{r\sin\theta} & \frac{f^{\phi\theta}}{r^2\sin\theta} & \frac{f^{\phi\phi}}{r^2\sin^2\theta} \end{bmatrix} + \begin{bmatrix} \mathfrak{h}^{tt} & 0 & 0 & 0 \\ 0 & \mathfrak{h}^{rr} & 0 & 0 \\ 0 & 0 & \frac{\mathfrak{h}^{\omega\omega}}{r^2} & 0 \\ 0 & 0 & 0 & \frac{\mathfrak{h}^{\omega\omega}}{r^2\sin^2\theta} \end{bmatrix}$$

where $\mathfrak{f}^{\gamma\delta} \in \ell^1 S(r^{-\kappa})$ and $\mathfrak{h}^{\gamma\delta} \in S_{rad}(r^{-\kappa})$. Thus Assumption 3 still holds with the added property $\mathfrak{h}^{tr} = 0$, as desired.

Next we achieve $\mathfrak{h}^{rr} = -\mathfrak{h}^{tt}$ via the coordinate change

$$d\rho = \left(1 + \chi_{>R}\frac{-\mathfrak{h}^{tt} - \mathfrak{h}^{rr}}{1 + \mathfrak{h}^{rr}}\right)^{\frac{1}{2}} dr$$

where R is a constant chosen so $1 + \mathfrak{h}^{rr} \gtrsim 1$ for $r > R$. We define

$$q_1(r) := \left(1 + \chi_{>R}\frac{-\mathfrak{h}^{tt} - \mathfrak{h}^{rr}}{1 + \mathfrak{h}^{rr}}\right)^{\frac{1}{2}}.$$

Since $\mathfrak{h}^{tt}, \mathfrak{h}^{rr} \in S_{rad}(r^{-\kappa})$, and $\chi_{>R}\frac{1}{1+\mathfrak{h}^{rr}} \lesssim 1$, we see $q_1^2 - 1 \in S_{rad}(r^{-\kappa})$ and $q_1 \in S_{rad}(1)$.

The coordinate change does not depend on t, so the $t = \textit{constant}$ subspaces are invariant under the change of coordinates and thus remain positive definite (i.e. space-like) and the metric coefficients remain independent of t, so Assumptions 1 and 2 still hold.

We now calculate $\mathfrak{g}^{\gamma\delta}$ in the new coordinate system. Since t, θ, and ϕ are unchanged, we need only calculate $\mathfrak{g}^{\rho\gamma}$ for $\gamma \in \{t, \rho, \theta, \phi\}$. First we calculate $\mathfrak{g}^{\rho t}$ and find

$$\begin{aligned}
\mathfrak{g}^{\rho\rho} &= q_1^2 \mathfrak{g}^{rr} \\
&= \Big(1 + \chi_{>R}\frac{-\mathfrak{h}^{tt} - \mathfrak{h}^{rr}}{1 + \mathfrak{h}^{rr}}\Big)(1 + \mathfrak{f}^{rr} + \mathfrak{h}^{rr}) \\
&= 1 + \mathfrak{f}^{rr} + \mathfrak{h}^{rr}(1 - \chi_{>R}) + \mathfrak{h}^{tt}(1 - \chi_{>R}) + \chi_{>R}\Big(\frac{-\mathfrak{h}^{tt} - \mathfrak{h}^{rr}}{1 + \mathfrak{h}^{rr}}\Big)\mathfrak{f}^{rr} - \mathfrak{h}^{tt} \\
&= 1 + \mathfrak{f}^{rr} + \chi_{<R}(\mathfrak{h}^{rr} + \mathfrak{h}^{tt}) + (q_1^2 - 1)\mathfrak{f}^{rr} - \mathfrak{h}^{tt}.
\end{aligned}$$

Note $\mathfrak{f}^{rr} \in \ell^1 S(r^{-\kappa})$, $\chi_{<R}$ is compactly supported, $\mathfrak{h}^{tt}, \mathfrak{h}^{rr} \in S_{rad}(r^{-\kappa})$, and $q_1^2 - 1 \in S_{rad}(r^{-\kappa})$. Thus we define

$$\tilde{\mathfrak{f}}^{\rho\rho} := \mathfrak{f}^{rr} + \chi_{<R}(\mathfrak{h}^{rr} + \mathfrak{h}^{tt}) + (q_1^2 - 1)\mathfrak{f}^{rr}$$

so that

$$\mathfrak{g}^{\rho\rho} = 1 + \tilde{\mathfrak{f}}^{\rho\rho} - \mathfrak{h}^{tt} \tag{3.5}$$

where $\tilde{\mathfrak{f}}^{\rho\rho} \in \ell^1 S(r^{-\kappa})$.

Next we calculate $\mathfrak{g}^{\rho t}$, $\mathfrak{g}^{\rho\theta}$, and $\mathfrak{g}^{\rho\phi}$. Since $\mathfrak{m}^{\gamma\delta}$ and $\mathfrak{h}^{\gamma\delta}$ are both diagonal matrices after the first coordinate change, we easily find

$$\begin{aligned}
\mathfrak{g}^{\rho t} &= q_1(r)\mathfrak{f}^{rt} := \tilde{\mathfrak{f}}^{\rho t} \\
\mathfrak{g}^{\rho\theta} &= q_1(r)\frac{\mathfrak{f}^{r\theta}}{r} =: \frac{\tilde{\mathfrak{f}}^{\rho\theta}}{r} \\
\mathfrak{g}^{\rho\phi} &= q_1(r)\frac{\mathfrak{f}^{r\phi}}{r\sin\theta} =: \frac{\tilde{\mathfrak{f}}^{\rho\phi}}{r\sin\theta}
\end{aligned} \tag{3.6}$$

where $\mathfrak{f}^{\rho\delta} \in \ell^1 S(r^{-\kappa})$ for $\delta \in \{t, \theta, \phi\}$ since $q_1 \in S_{rad}(1)$.

Combining (3.5) and (3.6) and relabeling yields

$$\left[\mathfrak{g}^{\gamma\delta}\right] = \left[\mathfrak{m}^{\gamma\delta}\right] + \begin{bmatrix} \mathfrak{f}^{tt} & \mathfrak{f}^{tr} & \frac{f^{t\theta}}{r} & \frac{f^{t\phi}}{r\sin\theta} \\ \mathfrak{f}^{rt} & \mathfrak{f}^{rr} & \frac{f^{r\theta}}{r} & \frac{f^{r\phi}}{r\sin\theta} \\ \frac{\mathfrak{f}^{\theta t}}{r} & \frac{\mathfrak{f}^{\theta r}}{r} & \frac{f^{\theta\theta}}{r^2} & \frac{f^{\theta\phi}}{r^2\sin\theta} \\ \frac{\mathfrak{f}^{\phi t}}{r\sin\theta} & \frac{\mathfrak{f}^{\phi r}}{r\sin\theta} & \frac{f^{\phi\theta}}{r^2\sin\theta} & \frac{f^{\phi\phi}}{r^2\sin^2\theta} \end{bmatrix} + \begin{bmatrix} \mathfrak{h}^{tt} & 0 & 0 & 0 \\ 0 & -\mathfrak{h}^{tt} & 0 & 0 \\ 0 & 0 & \frac{\mathfrak{h}^{\omega\omega}}{r^2} & 0 \\ 0 & 0 & 0 & \frac{\mathfrak{h}^{\omega\omega}}{r^2\sin^2\theta} \end{bmatrix} \tag{3.7}$$

where $\mathfrak{f}^{\gamma\delta} \in \ell^1 S(r^{-\kappa})$ and $\mathfrak{h}^{\gamma\delta} \in S_{rad}(r^{-\kappa})$. Thus Assumption 3 still holds with the added property $\mathfrak{h}^{rr} = -\mathfrak{h}^{tt}$, as desired. $\square$

Proposition 3.2. *In normalized coordinates (as in Lemma 3.1), the operator $\Box_{\mathfrak{g}} + V$ can be replaced by an L^2 self-adjoint operator, which can be written as*

$$P = -\partial_t^2 + \Delta + \partial_t P^1 + P^2$$

where

$$P^1 = \partial_i p_1^i(x) + p_1^i \partial_i, \quad p_1^i(x) \in \ell^1 S(r^{-\kappa}) \tag{3.8}$$

and

$$\begin{aligned} &P^2 = \partial_i p_2^{ij}(x)\partial_j + p_2^{\omega}(r)\Delta_\omega + V_\ell + V_r, \\ &p_2^{ij} \in \ell^1 S(r^{-\kappa}); \quad V_\ell \in \ell^1 S(r^{-\kappa-2}); \quad p_2^{\omega}, V_r \in S_{rad}(r^{-\kappa-2}). \end{aligned} \tag{3.9}$$

Proof. In this proof we will work in standard coordinates. Thus we begin by rewriting (3.7) in standard coordinates as

$$\left[\mathfrak{g}^{\alpha\beta}\right] = \left[\mathfrak{m}^{\alpha\beta}\right] + \left[\mathfrak{f}^{\alpha\beta}\right] + \left[\mathfrak{h}^{\alpha\beta}\right]$$

where

$$\left[\mathfrak{m}^{\alpha\beta}\right] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \left[\mathfrak{f}^{\alpha\beta}\right] = \begin{bmatrix} \mathfrak{f}^{tt} & \mathfrak{f}^{t1} & \mathfrak{f}^{t2} & \mathfrak{f}^{t3} \\ \mathfrak{f}^{t1} & \mathfrak{f}^{11} & \mathfrak{f}^{12} & \mathfrak{f}^{13} \\ \mathfrak{f}^{t2} & \mathfrak{f}^{12} & \mathfrak{f}^{22} & \mathfrak{f}^{23} \\ \mathfrak{f}^{t3} & \mathfrak{f}^{13} & \mathfrak{f}^{23} & \mathfrak{f}^{33} \end{bmatrix},$$

and

$$\left[\mathfrak{h}^{\alpha\beta}\right] = \begin{bmatrix} \mathfrak{h}^{tt} & 0 & 0 & 0 \\ 0 & \mathfrak{h}^{\omega\omega} & 0 & 0 \\ 0 & 0 & \mathfrak{h}^{\omega\omega} & 0 \\ 0 & 0 & 0 & h^{\omega\omega} \end{bmatrix} - \frac{\mathfrak{h}^{tt}+\mathfrak{h}^{\omega\omega}}{r^2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & x_1^2 & x_1x_2 & x_1x_3 \\ 0 & x_1x_2 & x_2^2 & x_2x_3 \\ 0 & x_1x_3 & x_2x_3 & x_3^2 \end{bmatrix}$$

with $\mathfrak{f}^{\alpha\beta} \in \ell^1 S(r^{-\kappa})$ and $\mathfrak{h}^{\alpha\beta} \in S_{rad}(r^{-\kappa})$. The matrix $\left[\mathfrak{h}^{\alpha\beta}\right]$ can be found by writing

$$\left[\mathfrak{h}^{\gamma\delta}\right] = \begin{bmatrix} \mathfrak{h}^{tt} & 0 & 0 & 0 \\ 0 & \mathfrak{h}^{\omega\omega} & 0 & 0 \\ 0 & 0 & \mathfrak{h}^{\omega\omega} & 0 \\ 0 & 0 & 0 & \mathfrak{h}^{\omega\omega} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathfrak{h}^{\omega\omega}+\mathfrak{h}^{tt} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and converting to spherical coordinates.

To make the operator self-adjoint, we will use the conjugation

$$\Box_{\mathfrak{g}} + V \to |\mathfrak{g}|^{\frac{1}{4}}(\Box_{\mathfrak{g}} + V)|\mathfrak{g}|^{-\frac{1}{4}}$$

Where $|\mathfrak{g}| = |\det(\mathfrak{g})|$. To assure that the coefficient of ∂_t^2 is -1 we multiply by $(-\mathfrak{g}^{tt})^{-1}$. We split this multiplication by multiplying by $(-\mathfrak{g}^{tt})^{-\frac{1}{2}}$ on the left and right so the operator is self-adjoint. Thus we will replace $\Box_{\mathfrak{g}} + V$ by

$$P = |\mathfrak{g}|^{1/4}(-\mathfrak{g}^{tt})^{-1/2}(\Box_g + V)(-\mathfrak{g}^{tt})^{-1/2}|\mathfrak{g}|^{-1/4}.$$

Note $|\mathfrak{g}|-|\mathfrak{m}| \in \ell^1 S(r^{-\kappa})+S_{rad}(r^{-\kappa})$ and $\mathfrak{g}^{tt} = -1+\mathfrak{f}^{tt}+\mathfrak{h}^{tt}$ with $\mathfrak{f}^{tt} \in \ell^1 S(r^{-\kappa})$ and $\mathfrak{h}^{tt} \in S_{rad}(r^{-\kappa})$.

We will show that P is of the desired form. Recall

$$\Box_{\mathfrak{g}} = \frac{1}{\sqrt{|\mathfrak{g}|}}\partial_\alpha \sqrt{|\mathfrak{g}|}\mathfrak{g}^{\alpha\beta}\partial_\beta$$

and

$$V = V_\ell + V_r, \qquad V_\ell \in \ell^1 S(r^{-\kappa-2}), \quad V_r \in S_{rad}(r^{-\kappa-2}).$$

We find

$$
\begin{aligned}
P &= |\mathfrak{g}|^{1/4}(-\mathfrak{g}^{tt})^{-1/2}|\mathfrak{g}|^{-\frac{1}{2}}\partial_\alpha|\mathfrak{g}|^{\frac{1}{2}}\mathfrak{g}^{\alpha\beta}\partial_\beta(-\mathfrak{g}^{tt})^{-1/2}|\mathfrak{g}|^{-1/4}+(-\mathfrak{g}^{tt})^{-1}V \\
&= (-\mathfrak{g}^{tt})^{-1/2}|\mathfrak{g}|^{-1/4}\partial_\alpha|\mathfrak{g}|^{\frac{1}{2}}\mathfrak{g}^{\alpha\beta}\partial_\beta(-\mathfrak{g}^{tt})^{-1/2}|\mathfrak{g}|^{-1/4}+(-\mathfrak{g}^{tt})^{-1}V.
\end{aligned}
\tag{3.10}
$$

The last term in (3.10) is readily seen to be a scalar term of the form

$$
V_\ell + V_r, \qquad V_\ell \in \ell^1 S(r^{-\kappa-2}), \quad V_r \in S_{rad}(r^{-\kappa-2})
$$

using our assumptions on V and the fact that $\mathfrak{g}^{tt} = 1 + \mathfrak{f}^{tt} + \mathfrak{h}^{tt}$.

To handle the first term in (3.10) we define

$$
A := (-\mathfrak{g}^{tt})^{-1/2}|\mathfrak{g}|^{-1/4} \quad \text{and} \quad B^{\alpha\beta} := |\mathfrak{g}|^{\frac{1}{2}}\mathfrak{g}^{\alpha\beta} \tag{3.11}
$$

so

$$
(-\mathfrak{g}^{tt})^{-1/2}|\mathfrak{g}|^{-1/4}\partial_\alpha|\mathfrak{g}|^{\frac{1}{2}}\mathfrak{g}^{\alpha\beta}\partial_\beta(-\mathfrak{g}^{tt})^{-1/2}|\mathfrak{g}|^{-1/4} = A\partial_\alpha B^{\alpha\beta}\partial_\beta A
$$

and calculate

$$
\begin{aligned}
&A\partial_\alpha B^{\alpha\beta}\partial_\beta A \\
&\quad = \partial_\alpha A B^{\alpha\beta}\partial_\beta A + [A,\partial_\alpha]B^{\alpha\beta}\partial_\beta A \\
&\quad = \partial_\alpha A^2 B^{\alpha\beta}\partial_\beta + \partial_\alpha A B^{\alpha\beta}[\partial_\beta, A] + [A,\partial_\alpha]B^{\alpha\beta}A\partial_\beta + [A,\partial_\alpha]B^{\alpha\beta}[\partial_\beta, A] \\
&\quad = \partial_\alpha A^2 B^{\alpha\beta}\partial_\beta + A B^{\alpha\beta}[\partial_\beta, A]\partial_\alpha + [\partial_\alpha, A B^{\alpha\beta}[\partial_\beta, A]] + [A,\partial_\alpha]B^{\alpha\beta}A\partial_\beta + [A,\partial_\alpha]B^{\alpha\beta}[\partial_\beta, A] \\
&\quad = \partial_\alpha A^2 B^{\alpha\beta}\partial_\beta + A B^{\alpha\beta}[\partial_\beta, A]\partial_\alpha - A B^{\alpha\beta}[\partial_\alpha, A]\partial_\beta + [\partial_\alpha, A B^{\alpha\beta}[\partial_\beta, A]] + [A,\partial_\alpha]B^{\alpha\beta}[\partial_\beta, A].
\end{aligned}
$$

The second two terms cancel out after summation, so we are left with

$$
A\partial_\alpha B^{\alpha\beta}\partial_\beta A = \partial_\alpha A^2 B^{\alpha\beta}\partial_\beta + [\partial_\alpha, A B^{\alpha\beta}[\partial_\beta, A]] + [A,\partial_\alpha]B^{\alpha\beta}[\partial_\beta, A]. \tag{3.12}
$$

For the last two terms in (3.12) we calculate

$$\begin{aligned}
&[\partial_\alpha, AB^{\alpha\beta}[\partial_\beta, A]] + [A, \partial_\alpha]B^{\alpha\beta}[\partial_\beta, A] \\
&\quad = [\partial_\alpha, A]B^{\alpha\beta}[\partial_\beta, A] + A[\partial_\alpha, B^{\alpha\beta}[\partial_\beta, A]] - [\partial_\alpha, A]B^{\alpha\beta}[\partial_\beta, A] \\
&\quad = AB^{\alpha\beta}[\partial_\alpha, [\partial_\beta, A]] + A[\partial_\alpha, B^{\alpha\beta}][\partial_\beta, A].
\end{aligned}$$

We use the notation ρ_ℓ^q and ρ_r^q to denote functions that are in $\ell^1 S(r^q)$ and $S_{rad}(r^q)$, respectively. We allow the precise form of ρ_ℓ^q and ρ_r^q each time they appear. In this notation we have

$$-\mathfrak{g}^{tt} = 1 + \rho_\ell^{-\kappa} + \rho_r^{-\kappa} \tag{3.13}$$

and

$$|\mathfrak{g}| = 1 + \rho_\ell^{-\kappa} + \rho_r^{-\kappa}. \tag{3.14}$$

We find

$$[\partial_\alpha, A] = \frac{1}{2}(-\mathfrak{g}^{tt})^{-\frac{3}{2}}(\partial_\alpha \mathfrak{g}^{tt})|\mathfrak{g}|^{-\frac{1}{4}} - \frac{1}{4}(-\mathfrak{g}^{tt})^{-\frac{1}{2}}|\mathfrak{g}|^{-\frac{5}{4}}(\partial_\alpha|\mathfrak{g}|)$$

so by (3.13) and (3.14),

$$[\partial_\alpha, A] = \rho_\ell^{-\kappa-1} + \rho_r^{-\kappa-1}$$

which yields

$$[\partial_\alpha, [\partial_\beta, A]] = \rho_\ell^{-\kappa-2} + \rho_r^{-\kappa-1}.$$

Similarly we find

$$[\partial_\alpha, B^{\alpha\beta}] = \frac{1}{2}|\mathfrak{g}|^{-\frac{1}{2}}(\partial_\alpha|\mathfrak{g}|)\mathfrak{g}^{\alpha\beta} + |\mathfrak{g}|^{\frac{1}{2}}(\partial_\alpha \mathfrak{g}^{\alpha\beta})$$

so by (3.13) and (3.14),

$$[\partial_\alpha, B^{\alpha\beta}] = \rho_\ell^{-\kappa-1} + \rho_r^{-\kappa-1}.$$

It follows that the last two terms in (3.12) can be included in the scalar potential terms P.

It is left to show the term

$$\partial_\alpha A^2 B^{\alpha\beta}\partial_\beta = \partial_\alpha(-\mathfrak{g}^{tt})^{-1}\mathfrak{g}^{\alpha\beta}\partial_\beta$$

in (3.12) is in the desired form. First we consider when $\alpha = \beta = t$. Here we find

$$\partial_t (-\mathfrak{g}^{tt})^{-1} \mathfrak{g}^{tt} \partial_t = -\partial_t^2 \tag{3.15}$$

as desired.

Next we consider when either $\alpha = t$ or $\beta = t$. Note

$$(-\mathfrak{g}^{tt})^{-1} \mathfrak{g}^{ti} = \frac{\mathfrak{f}^{ti}}{1 - \mathfrak{f}^{tt} - \mathfrak{h}^{tt}} \in \ell^1 S(r^{-\kappa}), \qquad i \in \{1, 2, 3\}$$

so

$$\partial_t (-\mathfrak{g}^{tt})^{-1} \mathfrak{g}^{ti} \partial_i + \partial_i (-\mathfrak{g}^{tt})^{-1} \mathfrak{g}^{ti} \partial_t = \partial_t p_1^i \partial_i + \partial_i p_1^i \partial_t \tag{3.16}$$

with $p_1^i \in \ell^1 S(r^{-\kappa})$, as desired.

Finally we consider the terms where $\alpha, \beta \in \{1, 2, 3\}$. Since we are only considering spatial terms we use i, j instead of α, β. If $i = j$ we find

$$\begin{aligned}
(-\mathfrak{g}^{tt})^{-1} \mathfrak{g}^{ii} &= \frac{1 + \mathfrak{f}^{ii} + \mathfrak{h}^{\omega\omega}(1 - x_i^2 r^{-2}) - \mathfrak{h}^{tt} x_i^2 r^{-2}}{1 - \mathfrak{f}^{tt} - \mathfrak{h}^{tt}} \\
&= 1 + \frac{\mathfrak{f}^{ii} + \mathfrak{f}^{tt}}{1 - \mathfrak{f}^{tt} - \mathfrak{h}^{tt}} + \frac{(\mathfrak{h}^{tt} + \mathfrak{h}^{\omega\omega})(1 - x_i^2 r^{-2})}{1 - \mathfrak{f}^{tt} - \mathfrak{h}^{tt}} \\
&= 1 + \frac{\mathfrak{f}^{ii} + \mathfrak{f}^{tt} + (\mathfrak{h}^{tt} + \mathfrak{h}^{\omega\omega})(1 - x_i^2 r^{-2})(\mathfrak{f}^{tt} + \mathfrak{h}^{tt})}{1 - \mathfrak{f}^{tt} - \mathfrak{h}^{tt}} + (\mathfrak{h}^{tt} + \mathfrak{h}^{\omega\omega})(1 - x_i^2 r^{-2})
\end{aligned}$$

where the second term is in $\ell^1 S(r^{-\kappa})$. Thus we may take

$$p_2^{ii} := \frac{\mathfrak{f}^{ii} + \mathfrak{f}^{tt} + (\mathfrak{h}^{tt} + \mathfrak{h}^{\omega\omega})(1 - x_i^2 r^{-2})(\mathfrak{f}^{tt} + \mathfrak{h}^{tt})}{1 - \mathfrak{f}^{tt} - \mathfrak{h}^{tt}}$$

so we have

$$(-\mathfrak{g}^{tt})^{-1} \mathfrak{g}^{ii} = 1 + p_2^{ii} + (\mathfrak{h}^{tt} + \mathfrak{h}^{\omega\omega})(1 - x_i^2 r^{-2}) \tag{3.17}$$

where $p_2^{ii} \in \ell^1 S(r^{-\kappa})$. If $i \neq j$ we find

$$\begin{aligned}
(-\mathfrak{g}^{tt})^{-1} \mathfrak{g}^{ij} &= \frac{\mathfrak{f}^{ij} - (\mathfrak{h}^{tt} + \mathfrak{h}^{\omega\omega}) x_i x_j r^{-2}}{1 - \mathfrak{f}^{tt} - \mathfrak{h}^{tt}} \\
&= \frac{\mathfrak{f}^{ij} - (\mathfrak{f}^{tt} + \mathfrak{h}^{tt})(\mathfrak{h}^{tt} + \mathfrak{h}^{\omega\omega}) x_i x_j r^{-2}}{1 - \mathfrak{f}^{tt} - \mathfrak{h}^{tt}} - (\mathfrak{h}^{tt} + \mathfrak{h}^{\omega\omega}) x_i x_j r^{-2}
\end{aligned}$$

where the first term is in $\ell^1 S(r^{-\kappa})$. Thus we may take

$$p_2^{ij} := \frac{\mathfrak{f}^{ij} - (\mathfrak{f}^{tt} + \mathfrak{h}^{tt})(\mathfrak{h}^{tt} + \mathfrak{h}^{\omega\omega})x_i x_j r^{-2}}{1 - \mathfrak{f}^{tt} - \mathfrak{h}^{tt}}$$

so we have

$$(-\mathfrak{g}^{tt})^{-1}\mathfrak{g}^{ij} = p_2^{ij} - (\mathfrak{h}^{tt} + \mathfrak{h}^{\omega\omega})x_i x_j r^{-2}, \quad i \neq j \tag{3.18}$$

where $p_2^{ij} \in \ell^1 S(r^{-\kappa})$.

Combining (3.17) and (3.18) yields

$$(-\mathfrak{g}^{tt})^{-1}\mathfrak{g}^{ij} = \delta_{ij} + p_2^{ij} + (\mathfrak{h}^{tt} + \mathfrak{h}^{\omega\omega})(\delta_{ij} - x_i x_j r^{-2})$$

where $p_2^{ij} \in \ell^1 S(r^{-\kappa})$. Thus we find

$$\partial_i(-\mathfrak{g}^{tt})^{-1}\mathfrak{g}^{ij}\partial_j = \Delta + \partial_i p_2^{ij}\partial_j + \partial_i(\mathfrak{h}^{tt} + \mathfrak{h}^{\omega\omega})(\delta_{ij} - x_i x_j r^{-2})\partial_j.$$

The first two terms are in the desired form. For the last term we calculate

$$\begin{aligned}
&\partial_i(\mathfrak{h}^{tt} + \mathfrak{h}^{\omega\omega})(\delta_{ij} - x_i x_j r^{-2})\partial_j \\
&\quad = x_i r^{-1}(\partial_r(\mathfrak{h}^{tt} + \mathfrak{h}^{\omega\omega}))(\delta_{ij} - x_i x_j r^{-2})\partial_j + (\mathfrak{h}^{tt} + \mathfrak{h}^{\omega\omega})\partial_i(\delta_{ij} - x_i x_j r^{-2})\partial_j \\
&\quad = (\partial_r(\mathfrak{h}^{tt} + \mathfrak{h}^{\omega\omega}))(\delta_{ij} x_i r^{-1}\partial_j - x_i^2 x_j r^{-3}\partial_j) + (\mathfrak{h}^{tt} + \mathfrak{h}^{\omega\omega})\partial_i(\delta_{ij} - x_i x_j r^{-2})\partial_j \\
&\quad = (\mathfrak{h}^{tt} + \mathfrak{h}^{\omega\omega})r^{-2}\Delta_\omega.
\end{aligned} \tag{3.19}$$

The calculations establishing the last equality can be found in the Appendix (see Section A.4). Defining

$$p_2^{\omega} = (\mathfrak{h}^{tt} + \mathfrak{h}^{\omega\omega})r^{-2}$$

we find

$$\partial_i(-\mathfrak{g}^{tt})^{-1}\mathfrak{g}^{ij}\partial_j = \Delta + \partial_i p_2^{ij}\partial_j + p_2^{\omega}\Delta_\omega \tag{3.20}$$

where $p_2^{\omega} \in S_{rad}(r^{-\kappa-2})$.

Combining (3.15), (3.16), and (3.20) we see

$$\partial_\alpha(-\mathfrak{g}^{tt})^{-1}\mathfrak{g}^{\alpha\beta}\partial_\beta = -\partial_t^2 + \Delta + \partial_t(\partial_i p_1^i + p_1^i\partial_i) + \partial_i p_2^{ij}\partial_j + p_2^\omega\Delta_\omega$$

with

$$p_1^i, p_2^{ij} \in \ell^1 S(r^{-\kappa}), \quad p_2^\omega \in S_{rad}(r^{-\kappa-2}).$$

We already showed

$$P = \partial_\alpha(-\mathfrak{g}^{tt})^{-1}\mathfrak{g}^{\alpha\beta}\partial_\beta + V_\ell + V_r$$

where $V_\ell \in \ell^1 S(r^{-\kappa-2})$ and $V_r \in S_{rad}(r^{-\kappa-2})$. Thus this concludes the proof. $\square$

3.1 Summary

Our goal is to prove pointwise bounds on u where u solves (1.6). The results of this section mean we now consider solutions u to the Cauchy problem

$$Pu(t,x) = 0 \qquad u(0,\cdot) = u_0 \qquad \partial_t u(0,\cdot) = u_1$$

where P is of the form

$$P = -\partial_t^2 + \Delta + \partial_t P^1 + P^2$$

with P^1 as in (3.8) and P^2 as in (3.9) with $\kappa \geq 2$. Recall κ indicates the rate at which the background geometry tends toward flat. Furthermore, the evolution satisfies the uniform energy estimate (1.4) and the weak local energy estimate (1.12) since these estimates are coordinate independent.

Therefore the results of Chapter 2 apply for all values of κ we consider. In particular, we have for $\Im\tau < 0$

$$\hat{u}(\tau) = R_\tau(-i\tau u_0 + P^1 u_0 - u_1)$$

so

$$u(t,x) = \frac{1}{\sqrt{2\pi}}\int_{\Im\tau=-\epsilon} R_\tau(-i\tau u_0 + P^1 u_0 - u_1)e^{it\tau}\, d\tau$$

for $\epsilon > 0$. Propositions 2.6 and 2.8 tell us R_τ extends continuously to $\tau \in \mathbb{R}$, so we may take the limit as $\epsilon \to 0$ to obtain

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} R_\tau(-i\tau u_0 + P^1 u_0 - u_1) e^{it\tau} \, d\tau. \tag{3.21}$$

In order to establish the pointwise bounds on $u(t,x)$, we will prove pointwise bounds on the resolvent in Chapter 5. These follow from the L^2 based bounds established in Chapter 2 combined with Sobolev embeddings.

Our argument will differ for $|\tau| \lesssim 1$ and $|\tau| \gtrsim 1$. Therefore we define

$$u_{>1}(t,x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \chi_{<1}(|\tau|) R_\tau(-i\tau u_0 + P^1 u_0 - u_1) e^{it\tau} \, d\tau$$

and

$$u_{<1}(t,x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \chi_{>1}(|\tau|) R_\tau(-i\tau u_0 + P^1 u_0 - u_1) e^{it\tau} \, d\tau.$$

In the high frequency case, the pointwise bounds that we are able to establish using the results of Chapter 2 will suffice to obtain arbitrarily fast decay for $u_{>1}(t,x)$. The low frequency case requires more work. The goal of the next chapter is to estimate $R_\tau g \approx R_0 g e^{-i\tau\langle r\rangle}$, calculate the error, and plug the results into our equation for $u_{<1}(t,x)$. We will see the role of the rate at which the background geometry tends to flat (indicated by κ) comes into play in this low frequency analysis. The error we obtain ultimately depends on κ, and the bounds we find from inverting the time Fourier transform of these error terms dictates how quickly the solution u decays.

CHAPTER 4

Zero and low frequency resolvent analysis

The goal of this chapter is to further analyze the resolvent at low frequencies so that we can establish pointwise decay rates for the low frequency part of $u(t,x)$. Recall in the main theorem we assume u solves the homogeneous Cauchy problem with initial data $u(0,\cdot) = u_0 \in Z^{\nu+1,\kappa}$ and $\partial_t u(0,\cdot) = u_1 \in Z^{\nu,\kappa+1}$. Here ν is a sufficiently large constant depending on κ and κ indicates the rate at which the background geometry tends toward flat. The low frequency part of $u(t,x)$ is given by

$$\begin{aligned} u_{<1}(t,x) &:= \frac{1}{\sqrt{2\pi}} \int_{\tau\in\mathbb{R}} \chi_{<1}(|\tau|)\hat{u}(\tau)e^{it\tau}\ d\tau \\ &= \frac{1}{\sqrt{2\pi}} \int_{\tau\in\mathbb{R}} \chi_{<1} R_\tau(-i\tau u_0 + P^1 u_0 - u_1)e^{it\tau}\ d\tau. \end{aligned} \tag{4.1}$$

We will approximate $R_\tau g \approx R_0 g e^{-i\tau r}$ for small τ, calculate the error, and plug the result into the above equation in order to obtain the final decay rate. In order to calculate the error, we will first need to obtain an expansion of $R_0 g$ in powers of $\langle r\rangle^{-1}$. In our argument establishing an expansion of $R_0 g$ for large r we will find

$$(-\Delta)(\chi_{>R} R_0 g) = h + \chi_{>R/2} P^2(\chi_{>R} R_0 g)$$

where $\|h\|_{Z^{n,\lambda}} \lesssim \|g\|_{Z^{n+4,\lambda}}$. This motivates the first Lemma below, where we will find an expansion for $(-\Delta)^{-1} g$ where $g \in Z^{n,\lambda}$.

In Lemma 4.1 we will obtain an expansion of $(-\Delta)^{-1} g$ for $g \in Z^{n,\lambda}$. In Lemma 4.2 we will obtain an expansion of $(-\Delta)^{-1} g$ for $g \in S_{rad}(r^{-q})$ with $q \geq 2$. In Proposition 4.3 we will obtain the expansion of $R_0 g$ for $g \in Z^{n+4,\lambda}$. Lemma 4.4 provides a straightforward calculation that we will use in Proposition 4.5 where we finally obtain the error for the estimate $R_\tau g \approx (R_0 g)e^{-i\tau\langle r\rangle}$.

Lemma 4.1. *Let* $\mathfrak{g} \in Z^{n,\lambda}$ *with* $\lambda \in \mathbb{N}$ *and* n *sufficiently large. We have the following representation* *for* $(-\Delta)^{-1}g$*:*

$$(-\Delta)^{-1}g = \sum_{j=0}^{\lambda-2}\Big(c_j\cdot\nabla^j\langle r\rangle^{-1} + e_j(r)\cdot(\nabla^j\langle r\rangle^{-1})\langle r\rangle^{j-\lambda+1}\Big) + d(r)\cdot\nabla^{\lambda-1}\langle r\rangle^{-1} + q(x) \tag{4.2}$$

where the coefficients satisfy

$$\sum_{j=0}^{\lambda-2}|c_j| + \|e_j\|_{\ell^1S(1)} + \|d\|_{L^\infty} + \|S_r d\|_{\ell^1S(1)} + \|q\|_{Z^{n+2,\lambda-2}} \lesssim \|g\|_{Z^{n,\lambda}}. \tag{4.3}$$

For $\lambda = 1$ *we have only the last two terms in* (4.2)*.*

Proof. We begin by proving if $g \in \mathcal{LE}^*$, then $v = (-\Delta)^{-1}g$ can be expressed as in (4.2) where the following estimate holds

$$\sum_{j=0}^{\lambda-2}|c_j| + \|e_j\|_{\ell^1S(1)} + \|d\|_{L^\infty} + \|S_r d\|_{\ell^1S(1)} + \sum_{i\le 2}\|\langle r\rangle^{-2+\lambda+i}\nabla^i q\|_{\mathcal{LE}^*} \lesssim \|\langle r\rangle^\lambda g\|_{\mathcal{LE}^*}. \tag{4.4}$$

Once (4.4) is established, the desired form (4.3) follows by elliptic regularity arguments, which we provide at the end of the proof.

Using the kernel for the fundamental solution of the Laplacian in $\mathbb{R}^3$ we have

$$v(x) \cong \int g(y)\frac{1}{|x-y|}\,dy.$$

We wish to be able to bound the coefficients of the representation for $(-\Delta)^{-1}g$ by the size of $\langle r\rangle^\lambda g$ measured in $\mathcal{LE}^*$. The $\mathcal{LE}^*$ norm is defined by the behavior of g on dyadic regions A_m. Thus it is of use to decompose g into these dyadic regions. To this end, we define

$$g_m := \beta_{\approx m}(r)g \quad\text{ and }\quad v_m := (-\Delta)^{-1}g_m = \int g_m(y)\frac{1}{|x-y|}\,dy$$

so we have

$$v \cong \sum_{m\ge 0}\int g_m(y)\frac{1}{|x-y|}\,dy = \sum_{m\ge 0} v_m.$$

We further decompose each v_m into v_m^{low} and v_m^{high} as follows

$$\begin{aligned} v_m^{low} &:= \chi_{<m+2}(r) v_m \\ &= \chi_{<m+2}(|x|) \int \beta_{\approx m}(|y|) g(y) \frac{1}{|x-y|}\, dy \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} v_m^{high} &:= \chi_{>m+2}(r) v_m \\ &= \chi_{>m+2}(|x|) \int \beta_{\approx m}(|y|) g(y) \frac{1}{|x-y|}\, dy. \end{aligned} \tag{4.6}$$

Thus we have

$$v \cong \sum_{m \geq 0} v_m^{low} + v_m^{high}.$$

The v_m^{low} terms have the advantage of being restricted to a bounded region. The v_m^{high} pieces have the advantage that $\frac{1}{|x-y|}$ is smooth in the region of integration. This will allow us to obtain a Taylor series expansion of $\frac{1}{|x-y|}$.

We claim $\sum_{m \geq 0} v_m^{low}$ satisfies the bounds on $q(x)$ in (4.4) and thus can be included in the $q(x)$ term in the statement of the proposition (we will see that the v_m^{high} pieces also generate a term that will be included in $q(x)$). In other words, we aim to prove

$$\sum_{i \leq 2} \left\| \langle r \rangle^{-2+\lambda+i} \nabla^i \sum_{m \geq 0} v_m^{low} \right\|_{\mathcal{LE}^*} \lesssim \left\| \langle r \rangle^{\lambda} g \right\|_{\mathcal{LE}^*}. \tag{4.7}$$

To see this we begin by obtaining L^2 bounds on $(-\Delta) v_m^{low}$ for all $m \geq 0$. These bounds will then transferred to $\nabla^2 v_m^{low}$ using integration by parts. We calculate

$$\begin{aligned} |(-\Delta) v_m^{low}| &= \left| (-\Delta) \left(\chi_{<m+2}(|x|) \int g_m(y) \frac{1}{|x-y|}\, dy \right) \right| \\ &\lesssim |g_m| + 2^{-m} |\chi'_{<m+2}(|x|)| \int \left| \beta_{\approx m}(|y|) g(y) \right| \frac{1}{|x-y|^2}\, dy \\ &\quad + 2^{-2m} |\chi''_{<m+2}(|x|)| \int \left| \beta_{\approx m}(|y|) g(y) \right| \frac{1}{|x-y|}\, dy \\ &\lesssim |g_m| + 2^{-3m} |\chi'_{<m+2}(|x|)| \|\beta_{\approx m} g\|_{L^1} + 2^{-3m} |\chi''_{<m+2}(|x|)| \|\beta_{\approx m} g\|_{L^1} \\ &\lesssim |g_m| + 2^{-\frac{3m}{2}} |\chi'_{<m+2}(|x|)| \|\beta_{\approx m} g\|_{L^2} + 2^{-\frac{3m}{2}} |\chi''_{<m+2}(|x|)| \|\beta_{\approx m} g\|_{L^2}. \end{aligned}$$

Thus we find

$$\begin{aligned}\|\Delta v_m^{low}\|_{L^2} &\lesssim \|g_m\|_{L^2} + 2^{-\frac{3m}{2}}\|\chi'_{<m+2}\|_{L^2}\|\beta_{\approx m} g\|_{L^2} + 2^{-\frac{3m}{2}}\|\chi''_{<m+2}\|_{L^2}\|\beta_{\approx m} g\|_{L^2} \\ &\lesssim \|g_m\|_{L^2}\end{aligned}$$

Since $\nabla^j v_m^{low} \to \infty$ as $r \to \infty$, integration by parts gives

$$\|\nabla^2 v_m^{low}\|_{L^2} \lesssim \|g_m\|_{L^2}.$$

Next we take advantage of the fact that v_m^{low} is supported on a bounded region and use the Poincaré inequality to find

$$2^{-2m}\|v_m^{low}\|_{L^2} + 2^{-m}\|\nabla v_m^{low}\|_{L^2} + \|\nabla^2 v_m^{low}\|_{L^2} \lesssim \|g_m\|_{L^2}.$$

In other words, $\|\nabla^i v_m^{low}\|_{L^2} \lesssim 2^{m(2-i)}\|g\|_{L^2(A_m)}$ for $i = 0, 1, 2$. Since $\|\nabla^i v_m^{low}\|_{L^2(A_m)} \lesssim \|\nabla^i v_m^{low}\|_{L^2}$, these estimates allow us to bound the L^2 norm of v_m^{low} and its derivatives in the dyadic region A_m, which presents the most difficulty since here we have $|x| \approx |y|$.

Finally we are ready to prove (4.7). To help with our calculations, we note that if $|x| \approx 2^k$ and $|y| \approx 2^m$ with $k \le m-1$, we have

$$|x-y|^{-1} \lesssim 2^{-m}, \qquad \left|\nabla |x-y|^{-1}\right| \lesssim 2^{-2m}, \qquad \left|\nabla^2 |x-y|^{-1}\right| \lesssim 2^{-3m}. \tag{4.8}$$

Furthermore, if we again assume $k \le m-1$, then $\|\nabla^i v_m^{low}\|_{L^2(A_k)} = \|\nabla^i v_m\|_{L^2(A_k)}$. We use the Cauchy-Schwarz inequality to find

$$\begin{aligned}\|\nabla^i v_m\|_{L^2(A_k)} &= \left\|\nabla^i \int g_m(y)\frac{1}{|x-y|}\ dy\right\|_{L^2(A_k)} \\ &\lesssim 2^{\frac{3k}{2}} 2^{-m(i+1)} 2^{\frac{3m}{2}}\|g\|_{L^2(A_m)}, \quad i = 0, 1, 2.\end{aligned} \tag{4.9}$$

Now we use (4.8) and (4.9) to calculate

$$
\begin{aligned}
\|\langle r\rangle^{-2+\lambda+i}\nabla^i \sum_m v_m^{low}\|_{\mathcal{LE}^*} &= \sum_k 2^{\frac{k}{2}}\|\langle r\rangle^{-2+\lambda+i}\sum_m \nabla^i v_m^{low}\|_{L^2(A_k)} \\
&\lesssim \sum_m \sum_{k<m+2} 2^{\frac{k}{2}}2^{k(-2+\lambda+i)}\|\nabla^i v_m^{low}\|_{L^2(A_k)} \\
&\lesssim \sum_m \sum_{k<m} 2^{k(\lambda+i)}2^{-m(i+1)}2^{\frac{3m}{2}}\|g\|_{L^2(A_m)} \\
&\quad + \sum_m \sum_{k\approx m} 2^{\frac{m}{2}}2^{m(-2+\lambda+i)}\|\nabla^i v_m^{low}\|_{L_2(A_{\approx m})} \\
&\lesssim \sum_m 2^{\frac{m}{2}}2^{m\lambda}\|g\|_{L^2(A_{\approx m})} \\
&\lesssim \|\langle r\rangle^{\lambda} g\|_{\mathcal{LE}^*}
\end{aligned}
$$

for $i = 0, 1, 2$. This concludes the proof of (4.7).

Now we turn our attention to $\sum_m v_m^{high}$. We consider the Taylor series expansion for $\frac{1}{|x-y|}$. For each m we integrate over $|y| \approx 2^m$ with $|x| \geq 2^{m+2}$. Thus $\frac{1}{|x-y|}$ is smooth in the region of integration and Taylor's theorem applies. We define y^j as follows. If the n^{th} component of ∇^j is $\partial_{i_1}\partial_{i_2}\cdots\partial_{i_j}$ then the n^{th} component of y^j is $y_{i_1}y_{i_2}\cdots y_{i_j}$. Note that $|y^j| \lesssim |y|^j$. Using this notation, Taylor's theorem yields

$$
\frac{1}{|x-y|} = \sum_{j=0}^{\lambda-1} \frac{\nabla^j r^{-1}\cdot (y)^j}{j!} + R_\lambda^x(y)
$$

where $r = |x|$ and

$$
R_\lambda^x(y) = \lambda!^{-1}\nabla^\lambda\left(|x-ty|^{-1}\right)\cdot (y)^\lambda
$$

for some $t \in (0, 1)$. Therefore

$$
v_m^{high}(x) = \sum_{j=0}^{\lambda-1} j!^{-1}\chi_{>m+2}(r)\int g_m(y)(\nabla^j r^{-1})\cdot y^j\ dy + \chi_{>m+2}(r)\int g_m(y)R_\lambda^x(y)\ dy
$$

where $r = |x|$.

We claim the last term can be included in the remainder $q(x)$ after summing over m. In other words, we aim to prove

$$
\sum_{i\leq 2}\|\langle r\rangle^{-2+\lambda+i}\nabla^i\tilde{q}\|_{\mathcal{LE}^*} \lesssim \|\langle r\rangle^{\lambda} g\|_{\mathcal{LE}^*}, \tag{4.10}
$$

where $\tilde{q}$ is defined by

$$\tilde{q}_m := \chi_{>m+2}(r) \int g_m(y) R_\lambda^x(y)\ dy$$
$$\tilde{q} := \sum_m \tilde{q}_m.$$

Note $|\nabla^\lambda(|x|^{-1})| \lesssim |x|^{-\lambda-1}$ and $|x-ty|^{-1} \lesssim |x|^{-1}$ since $|x| \geq 2|y|$ and $t \in (0,1)$. It follows that

$$\left|R_\lambda^y(x)\right| \lesssim \frac{|y|^\lambda}{|x|^{\lambda+1}}. \tag{4.11}$$

A straightforward calculation yields

$$\|\langle r\rangle^{-2+\lambda}\tilde{q}\|_{\mathcal{LE}^*} \leq \sum_m \sum_{l>m+2} \|\tilde{q}_m\|_{L^\infty(A_l)} 2^{l\lambda}. \tag{4.12}$$

The bound (4.11) implies

$$\|\tilde{q}_m\|_{L^\infty(A_l)} \lesssim 2^{-l(\lambda+1)} \|r^\lambda g_m\|_{L^1} \lesssim 2^{(m-l)(\lambda+1)} \|\langle r\rangle^{\frac{1}{2}} g_m\|_{L^2} \tag{4.13}$$

for $l > m+2$. Combining (4.12) and (4.13) gives

$$\|\langle r\rangle^{-2+\lambda}\tilde{q}(x)\|_{\mathcal{LE}^*} \lesssim \|g\|_{\mathcal{LE}^*},$$

as desired. The estimates for $\nabla^i\tilde{q}$ follow analogously once we note $|\nabla^i R_\lambda^y(x)| \lesssim \frac{|y|^\lambda}{|x|^{\lambda+i+1}}$ and the proof of (4.10) is concluded.

Next we consider

$$\sum_m \sum_{j=0}^{\lambda-1} (j!)^{-1} \chi_{>m+2}(|x|) \int g_m(y)(\nabla^j |x|^{-1}) y^j\ dy.$$

We define

$$c_j := \sum_m (j!)^{-1} \int g_m(y) y^j\ dy, \qquad e_j(r) := \sum_m (j!)^{-1} \langle r\rangle^{\lambda-1-j} (-\chi_{<m+2}) \int g_m(y) y^j\ dy$$

$$d(r) := \sum_m ((\lambda-1)!)^{-1}\chi_{>m+2}(r)\int g_m(y)y^{\lambda-1}\,dy$$

so

$$\sum_m\sum_{j=0}^{\lambda-1}(j!)^{-1}\chi_{>m+2}(|x|)\int g_m(y)(\nabla^j|x|^{-1})y^j\,dy$$
$$=\sum_{j=0}^{\lambda-2}\Big(c_j\cdot\nabla^j\langle r\rangle^{-1}+e_j(r)\cdot(\nabla^j\langle r\rangle^{-1})\langle r\rangle^{j-\lambda+1}\Big)+d(r)\cdot\nabla^{\lambda-1}\langle r\rangle^{-1}.$$

The desired bounds

$$\sum_{j=0}^{\lambda-2}(|c_j|+\|e_j\|_{\ell^1S(1)})+\|d\|_{L^\infty}+\|S_r d\|_{\ell^1S(1)}\lesssim\|g\|_{Z^{n,\lambda}}$$

follow easily using the following inequality:

$$\|\langle r\rangle^p g\|_{L^1(A_m)}\lesssim\|\langle r\rangle^{p+\frac{3}{2}}g\|_{L^2(A_m)}.$$

The details are provided in the Appendix (see section A.5.1). This concludes the argument showing if $\langle r\rangle^\lambda g\in\mathcal{LE}^*$, then we have the representation (4.2) where the coefficients satisfy estimate (4.4).

It is left to show if $g\in Z^{n,\lambda}$, then $v=(-\Delta)^{-1}g$ can be written as in (4.2) where the coefficients satisfy estimate (4.3). Let $g\in Z^{n,\lambda}$. Then g satisfies $\langle r\rangle^\lambda g\in\mathcal{LE}^*$, so $v=(-\Delta)^{-1}g$ admits a representation as in (4.2) such that (4.4) holds. Therefore to prove (4.3) we need to show

$$\|q\|_{Z^{n+2,\lambda-2}}\lesssim\|g\|_{Z^{n,\lambda}}. \tag{4.14}$$

We will do this by using (4.4) to show

$$\|q\|_{Z^{2,\lambda-2}}\lesssim\|g\|_{\mathcal{LE}^*}=\|g\|_{Z^{0,\lambda}} \tag{4.15}$$

and

$$\|\Delta q\|_{Z^{n,\lambda}}\lesssim\|g\|_{Z^{n,\lambda}}. \tag{4.16}$$

Then elliptic arguments allow us to bound $\|\nabla^2 q\|_{Z^{n,\lambda}}$ by $\|g\|_{Z^{n,\lambda}}$.

By (4.4) we have

$$\sum_{i\leq 2} \|\langle r\rangle^{-2+\lambda+i}\nabla^i q\|_{\mathcal{LE}^*} \lesssim \|\langle r\rangle^{\lambda} g\|_{\mathcal{LE}^*} = \|g\|_{Z^{0,\lambda}}.$$

The estimate (4.15) follows immediately from the observation

$$|T^i\Omega^j S_r^k \phi| \lesssim \sum_{l=0}^{i+j+k} |\langle r\rangle^l \nabla^l \phi| \tag{4.17}$$

for any ϕ with sufficient differentiability. We note (4.17) also implies

$$\|q\|_{Z^{2,\lambda-2}} + \|\nabla^2 q\|_{Z^{n,\lambda}} \lesssim \|g\|_{Z^{n,\lambda}} \quad \Rightarrow \quad \|q\|_{Z^{n+2,\lambda-2}} \lesssim \|g\|_{Z^{n,\lambda}} \tag{4.18}$$

since by (4.17) we have

$$\begin{aligned}\|q\|_{Z^{n+2,\lambda-2}} &\lesssim \sup_{N\leq n+2} \sum_{l=0}^{N} \|\langle r\rangle^{\lambda-2}\langle r\rangle^l \nabla^l q\|_{\mathcal{LE}^*} \\ &\lesssim \|\nabla^2 q\|_{Z^{n,\lambda}} + \|q\|_{Z^{2,\lambda-2}}.\end{aligned}$$

Therefore showing $\|\nabla^2 q\|_{Z^{n,\lambda}} \lesssim \|g\|_{Z^{n,\lambda}}$ combined with (4.15) yields (4.14).

Next we write

$$-\Delta q = g + \Delta(v-q) \tag{4.19}$$

and see that to establish (4.16), we wish to bound $\Delta(v-q)$. We have

$$-\Delta(v-q) = -\Delta\left(\sum_{j=0}^{\lambda-2}\left(c_j\cdot\nabla^j\langle r\rangle^{-1} + e_j(r)\cdot(\nabla^j\langle r\rangle^{-1})\langle r\rangle^{j-\lambda+1}\right) + d(r)\cdot\nabla^{\lambda-1}\langle r\rangle^{-1}\right).$$

We claim $\|\Delta(v-q)\|_{Z^{n,\lambda}} \lesssim \|g\|_{Z^{0,\lambda}}$. Once we prove the claim, (4.19) yields (4.16). The claim follows from the inequalities

$$\|\Delta(c_j\nabla^j\langle r\rangle^{-1})\|_{Z^{n,\lambda}} \lesssim \|\langle r\rangle^{\lambda} g\|_{\mathcal{LE}^*} \tag{4.20}$$

$$\|\Delta(e_j(\nabla^j\langle r\rangle^{-1})\langle r\rangle^{j-\lambda+1})\|_{Z^{n,\lambda}} \lesssim \|\langle r\rangle^{\lambda} g\|_{\mathcal{LE}^*} \tag{4.21}$$

$$\|\Delta(d\nabla^{\lambda-1}\langle r\rangle^{-1})\|_{Z^{n,\lambda}} \lesssim \|\langle r\rangle^{\lambda} g\|_{\mathcal{LE}^*}, \tag{4.22}$$

which we prove in the Appendix (see section A.5.2).

Now that we have established (4.15) and (4.16), we turn our attention to $\nabla^2 q$ and calculate

$$\begin{aligned}\|\nabla^2 q\|_{Z^{n,\lambda}} &= \sup_{i+j+k\leq n}\sum_m 2^{\frac{m}{2}}2^{m\lambda}\|T^i\Omega^j S_r^k\nabla^2 q\|_{L^2(A_m)} \\ &\lesssim \sup_{i+j+k\leq n}\sum_m 2^{\frac{m}{2}}2^{m\lambda}\|T^i\Omega^j S_r^k\nabla^2(\chi_{\approx m}q)\|_{L^2(\mathbb{R}^3)} \\ &\approx \sup_{i+j+k\leq n}\sum_m 2^{\frac{m}{2}}2^{m\lambda}\|T^i\Omega^j S_r^k\Delta(\chi_{\approx m}q)\|_{L^2(\mathbb{R}^3)}.\end{aligned} \tag{4.23}$$

for all $n\geq 1$. The last line follows from

$$\|T^i\Omega^j S_r^k\nabla^2(\chi_{\approx m}q)\|_{L^2(\mathbb{R}^3)} \approx \|T^{\leq i}\Omega^{\leq j}S_r^{\leq k}\Delta(\chi_{\approx m}q)\|_{L^2(\mathbb{R}^3)},$$

which is obtained using the standard integration by parts argument along with the commutators

$$[\partial_i,\Omega]\in \text{ span }\{T\},\quad [\partial_i,S_r]\in \text{ span }\{T\}$$

and

$$[\Delta,\Omega]=[\Delta,T]=0,\qquad [\Delta,S_r]=2\Delta.$$

Next we handle the right hand side of (4.23) by calculating

$$\Delta(\chi_{\approx m}q)=\chi_{\approx m}\Delta q+(\Delta\chi_{\approx m})q+2\nabla\chi_{\approx m}\cdot\nabla q \tag{4.24}$$

and

$$\Delta\chi_{\approx m}=2^{-2m}\chi''\left(\frac{r}{2^m}\right)+2^{-m+1}r^{-1}\chi'\left(\frac{r}{2^m}\right),\qquad \nabla\chi_{\approx m}=2^{-m}\frac{x}{r}\beta'\left(\frac{r}{2^m}\right). \tag{4.25}$$

Combining (4.23), (4.24), and (4.25) we find

$$
\begin{aligned}
\|\nabla^2 q\|_{Z^{n,\lambda}} &\lesssim \sup_{i+j+k\le n} \sum_m 2^{m(\lambda+\frac{1}{2})} \Big(\|T^i \Omega^j S_r^k \chi_{\approx m} \Delta q\|_{L^2(\mathbb{R}^3)} + \|T^i \Omega^j S_r^k (\Delta \chi_{\approx m}) q\|_{L^2(\mathbb{R}^3)} \\
&\qquad + \|T^i \Omega^j S_r^k (\nabla \chi_{\approx m}) \cdot \nabla q\|_{L^2(\mathbb{R}^3)} \Big) \\
&\lesssim \sup_{i+j+k\le n} \sum_m 2^{m(\lambda+\frac{1}{2})} \Big(\|T^i \Omega^j S_r^k \chi_{\approx m} \Delta q\|_{L^2(\mathbb{R}^3)} + \|2^{-2m} T^i \Omega^j S_r^k \chi'' \left(\frac{r}{2^m}\right) q\|_{L^2(\mathbb{R}^3)} \\
&\qquad + 2^{-m} \|T^i \Omega^j S_r^k r^{-1} \chi' \left(\frac{r}{2^m}\right) q\|_{L^2(\mathbb{R}^3)} + 2^{-m} \|T^i \Omega^j S_r^k \chi' \left(\frac{r}{2^m}\right) \nabla q\|_{L^2(\mathbb{R}^3)} \Big) \\
&\lesssim \|\Delta q\|_{Z^{n,\lambda}} + \|\nabla q\|_{Z^{n,\lambda-1}} + \|q\|_{Z^{n,\lambda-2}}
\end{aligned}
\tag{4.26}
$$

for all n.

Finally we are ready to prove (4.14), which we do by induction. Set $n = 1$. By (4.15) we have

$$\|q\|_{Z^{1,\lambda-2}} \lesssim \|g\|_{Z^{0,\lambda}} \lesssim \|g\|_{Z^{1,\lambda}}$$

so the third term on the right hand side of (4.26) is controlled by $\|g\|_{Z^{1,\lambda}}$. For the first term on the right hand side of (4.26) we see by (4.16)

$$\|\Delta q\|_{Z^{1,\lambda}} \lesssim \|g\|_{Z^{1,\lambda}}.$$

For the second term on the right hand side of (4.26) we use Lemma A.4 and (4.15) to find

$$\|\nabla q\|_{Z^{1,\lambda-1}} \lesssim \|g\|_{Z^{0,\lambda}} \lesssim \|g\|_{Z^{n,\lambda}}.$$

Therefore we have

$$\|\nabla^2 q\|_{Z^{1,\lambda}} \lesssim \|g\|_{Z^{1,\lambda}}$$

which combined with (4.15) yields

$$\|q\|_{Z^{3,\lambda-2}} \lesssim \|g\|_{Z^{1,\lambda}}$$

by (4.18). Thus (4.14) holds for $n = 1$.

Now fix N and assume (4.14) holds for $n < N$. By (4.16) we have

$$\|\Delta q\|_{Z^{N,\lambda}} \lesssim \|g\|_{Z^{N,\lambda}}.$$

The inductive hypothesis gives

$$\|q\|_{Z^{N+1,\lambda-2}} \lesssim \|g\|_{Z^{N-1,\lambda}} \lesssim \|g\|_{Z^{N,\lambda}}$$

then applying Lemma A.4 yields

$$\|\nabla q\|_{Z^{N,\lambda-1}} \lesssim \|g\|_{Z^{N,\lambda}}.$$

Then we use (4.26) to find

$$\|\nabla^2 q\|_{Z^{N,\lambda}} \lesssim \|g\|_{Z^{N,\lambda}},$$

which combined with (4.15) gives

$$\|q\|_{Z^{N+2,\lambda-2}} \lesssim \|g\|_{Z^{N,\lambda}}$$

by (4.18), as desired. This concludes the proof of (4.14) and thus the proof of the proposition. □

In Propositions 4.3 and 4.5 we will need $(-\Delta)^{-1}g$ for $g \in S_{rad}(r^{-q})$ where $q \geq 2$. We provide these calculations in the following lemma.

Lemma 4.2. *Let $g \in S_{rad}(r^{-q})$ and set $v = (-\Delta)^{-1}g$*

1. *If $q \geq 4$ then for large r*

$$v = c\langle r\rangle^{-1} + e(r)\langle r\rangle^{-(q-2)}$$

for some constant c and $e \in S_{rad}(1)$.

2. *If $q = 3$ then*

$$v = \epsilon(r)\langle r\rangle^{-1}$$

for $\epsilon \in S_{rad}(\ln r)$.

3. *If $q = 2$ then*

$$v = \epsilon(r)$$

for $\epsilon \in S_{rad}(\ln r)$.

Proof. **1 and 2.** $q \geq 3$

Since $-\Delta v = g$, we can write

$$\partial_r^2(rv) = -rg. \tag{4.27}$$

Integrating (4.27) from infinity we find

$$-\partial_r(rv) = \int_r^\infty \partial_s^2(sv(s))\ ds = \int_r^\infty sg(s)\ ds. \tag{4.28}$$

For the right hand side of (4.28) we calculate

$$\begin{aligned} \left| \int_r^\infty sg(s)\ ds \right| &\leq \int_r^\infty \langle s \rangle^{-q+1}\ ds \\ &\equiv \langle r \rangle^{-q+2}. \end{aligned} \tag{4.29}$$

Note $\langle s \rangle^{-q+1}$ is integrable from infinity since we are in the case $q \geq 3$. By (4.28) and (4.29) we see

$$\partial_r(rv) \in S_{rad}(r^{-q+2}).$$

We define $g_1 := \partial_r(rv)$ and integrate from 0 to find

$$rv = \int_0^r \partial_s(sv(s))\ ds = \int_0^r g_1(s)\ ds$$

where $g_1 \in S_{rad}(r^{-q+2})$.

When $q \geq 4$ we have $-q+2 \leq -2$ so (4.30) yields

$$rv = c + g_2$$

where $g_2 \in S_{rad}(r^{-q+3})$. Taking

$$e(r) := g_2(r)\langle r \rangle^{q-3},$$

we find for large r

$$v = c\langle r \rangle^{-1} + e(r)\langle r \rangle^{-(q-2)}$$

for some constant c and $e \in S_{rad}(1)$, as desired.

When $q = 3$, we have (4.30) with $g_1 \in S_{rad}(r^{-1})$ which yields

$$rv = c + g_2$$

where $g_2 \in S_{rad}(\ln r)$. Note $c \in S_{rad}(1) \subseteq S_{rad}(\ln r)$. Thus taking

$$\epsilon(r) = c + g_2$$

we find for large r

$$v = \epsilon(r)\langle r\rangle^{-1}$$

where $\epsilon \in S_{rad}(\ln r)$, as desired.

2. $q = 2$

We write the equation $v = (-\Delta)^{-1}g$ as

$$v(r_0) = -\int_0^{r_0} s^{-2} \int_0^s r^2 g(r)\ dr\ ds. \tag{4.30}$$

where $g \in S_{rad}(r^{-2})$ by assumption. We find

$$\left|\int_0^s r^2 g(r)\ dr\right| \lesssim \int_0^s 1\ dr = s.$$

Therefore (4.30) becomes

$$v(r_0) = \int_0^{r_0} s^{-2} g_1(s)\ ds$$

where $g_1 \in S_{rad}(r)$ and we have

$$|v(r_0)| = \left|\int_0^{r_0} s^{-2} g_1(s)\ ds\right| \lesssim |\ln r_0|.$$

Thus $v(r) = \epsilon(r)$ with $\epsilon(r) \in S_{rad}(\ln r)$, as desired. $\square$

We now prove a direct analogue of Proposition 4.1 for $R_0 = (\Delta + P^2)^{-1}$ instead of Δ^{-1}.

Proposition 4.3. *Let $g \in Z^{n+4,\lambda}$ with n sufficiently large and $\lambda \in \mathbb{N}$. Take $v = R_0 g$.*

If $1 \le \lambda \le \kappa$, then for large r, v can be written as in (4.2) where the following estimate holds

$$\sum_{j=0}^{\lambda-2} \Big(|c_j| + \|e_j\|_{\ell^1 S(1)} \Big) + \|d\|_{L^\infty} + \|S_r d\|_{\ell^1 S(1)} + \|q\|_{Z^{n+2,\lambda-2}} \lesssim \|g\|_{Z^{n+4,\lambda}}. \tag{4.31}$$

If $\lambda = \kappa + 1$, then for large r, v can be written as in (4.2) where the following estimate holds

$$\sum_{j=0}^{\lambda-2} |c_j| + \|e_0\|_{S(1)} + \sum_{j=1}^{\lambda-2} \|e_j\|_{\ell^1 S(1)} + \|d\|_{L^\infty} + \|S_r d\|_{\ell^1 S(1)} + \|q\|_{Z^{n+2,\lambda-2}} \lesssim \|g\|_{Z^{n+4,\lambda}}. \tag{4.32}$$

Before proving the proposition we provide a brief summary of the argument. Since we are concerned only with large r, we consider $\chi_{>R} v =: w$ and show

$$-\Delta w = h + \chi_{>R/2} P^2 w$$

where $\|h\|_{Z^{n,\lambda}} \lesssim \|g\|_{Z^{n+4,\lambda}}$. We then use use Lemma 4.1 to obtain the desired form for w. For $\lambda \le \kappa$, this works by showing that the above equation for w is perturbative with respect to (4.3). The calculations for $\lambda \le \kappa$ apply to the $\lambda = \kappa + 1$ case, but one term which fails to be perturbative needs to be dealt with. The remaining term has the benefit of being radial and will be handled using Lemma 4.2.

Proof. Set $w = \chi_{>R} v = \chi_{>R} R_0 g$ where $\chi_{>R}(r) = \chi_{>}(\frac{r}{R})$. Since $w = v$ for large r, any expression we obtain for w holds for v when r is large.

We write

$$P_0 w = \chi_{>R} g + [P_0, \chi_{>R}] v =: h. \tag{4.33}$$

Recall the operator P_τ is given by

$$P_\tau = \Delta + \tau^2 + i\tau P^1 + P^2$$

where

$$P^1 = \partial_i p_1^i + p_1^i \partial_i, \qquad p_1^i \in \ell^1 S(r^{-\kappa})$$

and

$$P^2 = \partial_i p_2^{ij} \partial_j + p_2^\omega \Delta_\omega + V_\ell + V_r$$

$$p_2^{ij} \in \ell^1 S(r^{-\kappa}), \qquad p_2^\omega, V_r \in S_{rad}(r^{-\kappa-2}), \quad \text{and} \quad V_\ell \in \ell^1 S(r^{-\kappa-2}).$$

Thus $P_0 = \Delta + P^2$, and we calculate

$$\begin{aligned}
[P_0, \chi_{>R}(r)] &= [\Delta + \partial_i p_2^{ij} \partial_j + p_2^\omega \Delta_\omega + V_\ell + V_r, \chi_{>R}(r)] \\
&= (\Delta \chi_{>R}) + 2(\nabla \chi_{>R}) \cdot \nabla + [p_2^{ij} \partial_i \partial_j, \chi_{>R}] + [(\partial_i p_2^{ij}) \partial_j, \chi_{>R}] \\
&= (\Delta \chi_{>R}) + 2(\nabla \chi_{>R}) \cdot \nabla + p_2^{ij} (\partial_i \partial_j \chi_{>R}) + p_2^{ij} (\partial_i \chi_{>R}) \partial_j + p_2^{ij} (\partial_j \chi_{>R}) \partial_i \\
&\quad + (\partial_i p_2^{ij})(\partial_j \chi_{>R}).
\end{aligned}$$

Since $\chi_{>R} \equiv 0$ for $r < R$ and $\chi_{>R} \equiv 1$ for $r > 2R$, we see the commutator $[P_0, \chi_{>R}]$ is supported on $[R, 2R]$. Thus (4.33) gives

$$\begin{aligned}
\|\langle r \rangle^\lambda [P_0, \chi_{>R}] v\|_{\mathcal{LE}^*} &\lesssim \sum_{m=\log R}^{\log 2R} \Big[\|\langle r \rangle^{\frac{1}{2}+\lambda} (R^{-2} + R^{-1} \rho_\ell^{-\kappa-1} + R^{-2} \rho_\ell^{-\kappa}) v\|_{L^2(A_m)} \\
&\qquad + \|\langle r \rangle^{\frac{1}{2}+\lambda} (R^{-1} + R^{-1} \rho_\ell^{-\kappa}) \nabla v\|_{L^2(A_m)} \Big] \\
&\lesssim_R \|\langle r \rangle^{-1} v\|_{\mathcal{LE}} + \|\nabla v\|_{\mathcal{LE}} \\
&\lesssim \|v\|_{\mathcal{LE}_0}
\end{aligned} \tag{4.34}$$

where

$$\|v\|_{\mathcal{LE}_0} = \|\langle r \rangle^{-1} v\|_{\mathcal{LE}} + \|\nabla v\|_{\mathcal{LE}} + \|\langle r \rangle \nabla^2 v\|_{\mathcal{LE}}$$

by the definition of the $\mathcal{LE}_\tau$ norm in (2.8). The constants in the inequalities in (4.34) depend on R, but this is not an issue since R is fixed. Now (4.33) and (4.34) yield

$$\|\langle r \rangle^\lambda h\|_{\mathcal{LE}^*} \lesssim \|\langle r \rangle^\lambda g\|_{\mathcal{LE}^*,} + \|v\|_{\mathcal{LE}_0} \lesssim \|\langle r \rangle^\lambda g\|_{\mathcal{LE}^{*,4}}$$

since $\|v\|_{\mathcal{LE}_0} \lesssim \|\langle r \rangle g\|_{\mathcal{LE}^{*,4}}$ by Proposition 2.8.

Similarly we find

$$\begin{aligned}
\|h\|_{Z^{n,\lambda}} &\lesssim \|g\|_{Z^{n,\lambda}} + \sup_{i+j+k\le n} \|\langle r\rangle^\lambda T^i \Omega^j S_r^k [P_0, \chi_{>R}] v\|_{\mathcal{LE}^*} \\
&\lesssim \|g\|_{Z^{n,\lambda}} + \sup_{i+j+k\le n} \|\langle r\rangle^\lambda [P_0, \chi_{>R}] T^i \Omega^j S_r^k v\|_{\mathcal{LE}^*} + \|\langle r\rangle^\lambda [T^i \Omega^j S_r^k, \langle r\rangle^\lambda [P_0, \chi_{>R}]] v\|_{\mathcal{LE}^*} \\
&\lesssim_R \|g\|_{Z^{n,\lambda}} + \sup_{i\le n} \|\langle r\rangle^\lambda T^i v\|_{\mathcal{LE}^*([R,2R])} \\
&\lesssim_R \|g\|_{Z^{n,\lambda}} + \|v\|_{\mathcal{LE}_0^n}
\end{aligned}$$

where $\mathcal{LE}^*([R,2R])$ indicates the $\mathcal{LE}^*$ norm restricted to $R \le r \le 2R$. The last inequality follows due to the commutator $[P_0, \chi_{>R}]$ being supported on $[R, 2R]$ and (4.17). Thus by Proposition 2.8 we have

$$\|h\|_{Z^{n,\lambda}} \lesssim \|g\|_{Z^{n+4,\lambda}} \tag{4.35}$$

where we allow the constant in the inequality to depend on R.

Rewriting the equation $P_0 w = h$ using $P_0 = \Delta + P^2$, we obtain

$$-\Delta w = -h + \chi_{>R/2} P^2 w. \tag{4.36}$$

Note $\chi_{>R/2} P^2 w = P^2 w$ because the support of $P^2 w$ is contained in the region where $\chi_{>R/2} = 1$.

By Lemma 4.1, w can be written as in (4.2) where (4.3) holds so we have

$$\begin{aligned}
&\sum_{j=0}^{\lambda-2} |c_j| + \|e_j\|_{\ell^1 S(1)} + \|d(r)\|_{L^\infty} + \|S_r d\|_{\ell^1 S(1)} + \|q\|_{Z^{n+2,\lambda-2}} \\
&\qquad \le C \left\| \chi_{>\frac{R}{2}} P^2 \left(\sum_{j=0}^{\lambda-2} \Big(c_j \cdot \nabla^j \langle r\rangle^{-1} + e_j \cdot (\nabla^j \langle r\rangle^{-1}) \langle r\rangle^{j-\lambda+1} \Big) + d \cdot \nabla^{\lambda-1} \langle r\rangle^{-1} + q \right) \right\|_{Z^{n,\lambda}} \\
&\qquad\quad + C_R \|h\|_{Z^{n,\lambda}}.
\end{aligned} \tag{4.37}$$

If we can show

$$\begin{aligned}\left\|\chi_{>\frac{R}{2}}P^2\left(\sum_{j=0}^{\lambda-2}\left(c_j\cdot\nabla^j\langle r\rangle^{-1}+e_j\cdot(\nabla^j\langle r\rangle^{-1})\langle r\rangle^{j-\lambda+1}\right)+d\cdot\nabla^{\lambda-1}\langle r\rangle^{-1}+q\right)\right\|_{Z^{n,\lambda}}\\ \lesssim R^{-1}\left(\sum_{j=0}^{\lambda-2}|c_j|+\|e_j\|_{\ell^1S(1)}+\|d\|_{L^\infty}+\|S_rd\|_{\ell^1S(1)}+\|q\|_{Z^{n+2,\lambda-2}}\right)\end{aligned} \tag{4.38}$$

then choosing R sufficiently large allows us to bootstrap the perturbative term in (4.38) to obtain

$$\begin{aligned}\sum_{j=0}^{\lambda-2}(|c_j|+\|e_j\|_{\ell^1S(1)})+\|d(r)\|_{L^\infty}+\|S_rd\|_{\ell^1S(1)}+\|q\|_{Z^{n+2,\lambda-2}} &\le (1-cR^{-1})^{-1}C_R\|\langle r\rangle^\lambda h\|_{Z^{n,\lambda}}\\ &\le (1-cR^{-1})^{-1}C_R\|g\|_{Z^{n+4,\lambda}}\end{aligned}$$

as desired. Note the last inequality comes from (4.35). Thus we wish to show

$$\|\chi_{>\frac{R}{2}}P^2c_j\cdot\nabla^j\langle r\rangle^{-1}\|_{Z^{n,\lambda}}\lesssim R^{-1}|c_j|,\qquad 0\le j\le\lambda-2 \tag{4.39}$$

$$\|\chi_{>\frac{R}{2}}P^2e_j\cdot(\nabla^j\langle r\rangle^{-1})\langle r\rangle^{j-\lambda+1}\|_{Z^{n,\lambda}}\lesssim R^{-1}\|e_j\|_{\ell^1S(1)},\qquad 1\le j\le\lambda-2 \tag{4.40}$$

$$\|\chi_{>\frac{R}{2}}P^2d\nabla^{\lambda-1}\langle r\rangle^{-1}\|_{Z^{n,\lambda}}\lesssim R^{-1}(\|d\|_{L^\infty}+\|S_rd\|_{\ell^1S(1)}) \tag{4.41}$$

$$\|\chi_{>\frac{R}{2}}P^2q\|_{Z^{n,\lambda}}\lesssim R^{-1}\|q\|_{Z^{n+2,\lambda-2}}. \tag{4.42}$$

We will see (4.40), (4.41), and (4.42) hold for $\lambda\le\kappa+1$. And we will see (4.39) holds for $\lambda\le\kappa+1$ when $1\le j\le\lambda-2$. When $j=0$, (4.39) holds only for $\lambda\le\kappa$. Our argument handling $c_0\langle r\rangle^{-1}$ when $\lambda=\kappa+1$ will change the space we can assume e_0 is in, causing the difference in the result for $\lambda=\kappa+1$.

First we prove (4.42). Direct calculation yields

$$\begin{aligned}\|\langle r\rangle^\lambda \chi_{>R/2} P^2 q\|_{\mathcal{LE}^*} &\lesssim \sum_{a\le 2} \|\chi_{>R/2}\langle r\rangle^\lambda \rho_\ell^{-\kappa-2+a}\nabla^a q\|_{\mathcal{LE}^*} + \|\chi_{>R/2}\langle r\rangle^\lambda \rho_r^{-\kappa-2}\Delta_\omega q\|_{\mathcal{LE}^*}\\ &\quad + \|\chi_{>R/2}\langle r\rangle^\lambda \rho_r^{-\kappa-2} q\|_{\mathcal{LE}^*}\\ &\lesssim \sum_{a\le 2} \|\chi_{>R/2}\langle r\rangle^{\lambda-\kappa-2+a}\nabla^a q\|_{\mathcal{LE}^*} + \|\chi_{>R/2}\langle r\rangle^{\lambda-\kappa-2}(r^2\Delta - r^2\partial_r^2 - 2r\partial_r) q\|_{\mathcal{LE}^*}\\ &\lesssim \sum_{a\le 2} \|\chi_{>R/2}\langle r\rangle^{\lambda-\kappa-2+a}\nabla^a q\|_{\mathcal{LE}^*},\end{aligned} \tag{4.43}$$

and we note (4.43) holds for any q with enough regularity (thus we can use (4.43) in our calculations for c_j, e_j, and d). Now we have

$$\begin{aligned}&\|\langle r\rangle^\lambda T^i \Omega^j S_r^k \chi_{>R/2} P^2 q\|_{\mathcal{LE}^*}\\ &\quad\lesssim \sum_{a\le 2} \|\langle r\rangle^{\lambda-\kappa-2+a}\nabla^a T^i\Omega^j S_r^k q\|_{\mathcal{LE}^*([\frac{R}{2},\infty))} + \|\langle r\rangle^\lambda [T^i\Omega^j S_r^k, \chi_{>R/2}P^2] q\|_{\mathcal{LE}^*}.\end{aligned} \tag{4.44}$$

For the commutator $[T^i\Omega^j S_r^k, \chi_{>R/2}P^2]$ we find

$$[T^i\Omega^j S_r^k, \chi_{>R/2}P^2] = [T^i, \chi_{>R/2}]\Omega^j S_r^k P^2 + T^i\Omega^j [S_r^k, \chi_{>R/2}] P^2 + \chi_{>R/2}[T^i\Omega^j S_r^k, P^2].$$

The commutators $[S_r^k, \chi_{>R/2}]$ and $[T^i, \chi_{>R/2}]$ are compactly supported and uniformly bounded in R. We note

$$[\Gamma, P^2] = Q^2$$

for some Q^2 of the form

$$Q^2 = \partial_i h^{ij}\partial_j + h^\omega \Delta_\omega + h_r + h_\ell, \qquad h^{ij} \in \ell^1 S(r^{-\kappa}), \quad h^\omega, h_r \in S_{rad}(r^{-\kappa-2}), \quad \text{and } h_\ell \in \ell^1 S(r^{-\kappa-2}),$$

and

$$[\Gamma, Q^2] = Q^2$$

where we allow the precise form of Q^2 to change each time it appears. It follows that $[\Gamma^n, P^2] = Q^2\Gamma^{<n}$. We note P^2 is an operator with the same form as Q^2 and the calculations above are the same as

those used in Proposition 2.5 to calculate $[P_\tau, T^i\Omega^j S^k]$. Now we can use (4.43) to obtain

$$\sup_{i+j+k\le n} \|\langle r\rangle^\lambda [T^i\Omega^j S_r^k, \chi_{>R/2}P^2]q\|_{\mathcal{LE}^*} \lesssim \sup_{i+j+k\le n-1} \sum_{a\le 2} \|\langle r\rangle^{\lambda-\kappa-2+a}\nabla^a T^i\Omega^j S_r^k q\|_{\mathcal{LE}^*([\frac{R}{2},\infty))},$$

which combined with (4.44) yields

$$\begin{aligned} \|\chi_{>R/2}P^2 q\|_{Z^{n,\lambda}} &\lesssim \|q\|_{Z^{n+2,\lambda-\kappa-2}([\frac{R}{2},\infty))} \\ &\lesssim R^{-1}\|q\|_{Z^{n+2,\lambda-2}}, \end{aligned} \tag{4.45}$$

using $|\nabla\phi| \le r^{-1}(|S_r\phi| + |\Omega\phi|)$ for general ϕ, as desired.

To prove (4.41) we use the notation $\rho^q \in S(r^q)$ but allow ρ^q to indicate different functions in $S(r^q)$ each time it appears. Using this notation and (4.45) we find

$$\begin{aligned} \|\chi_{>R/2}P^2 d(r)\nabla^{\lambda-1}\langle r\rangle^{-1}\|_{Z^{n,\lambda}} &\lesssim \|\chi_{>R/2}P^2 d(r)\rho^{-\lambda}\|_{Z^{n,\lambda}} \\ &= \|d(r)\rho^{-\lambda}\|_{Z^{n+2,\lambda-\kappa-2}([\frac{R}{2},\infty))} \\ &\lesssim \|S_r(d\rho^{-\lambda})\|_{Z^{n+1,\lambda-\kappa-2}([\frac{R}{2},\infty))} + \|\Omega(d\rho^{-\lambda})\|_{Z^{n+1,\lambda-\kappa-2}([\frac{R}{2},\infty))} \\ &\qquad + \|T(d\rho^{-\lambda})\|_{Z^{n+1,\lambda-\kappa-2}([\frac{R}{2},\infty))} + \|d\rho^{-\lambda}\|_{Z^{0,\lambda-\kappa-2}([\frac{R}{2},\infty))} \\ &\lesssim \|(S_r d)\rho^{-\lambda}\|_{Z^{n+1,\lambda-\kappa-2}([\frac{R}{2},\infty))} + \|d\rho^{-\lambda}\|_{Z^{n+1,\lambda-\kappa-2}([\frac{R}{2},\infty))} \\ &\lesssim \sum_{b=0}^{n+1} \|(S_r d)\rho^{-\lambda}\|_{Z^{b,\lambda-\kappa-2}([\frac{R}{2},\infty))} + \|d\rho^{-\lambda}\|_{Z^{0,\lambda-\kappa-2}([\frac{R}{2},\infty))} \\ &\lesssim R^{-1}\|S_r d\|_{\ell^1 S(1)} + R^{-1}\|d\|_{L^\infty}. \end{aligned}$$

The last inequality follows from Lemma A.2 and Lemma A.3.

To prove (4.40) we use (4.42) and Lemma A.3 to find

$$\|\chi_{>R/2}P^2 e_j(\nabla^j\langle r\rangle^{-1})\langle r\rangle^{j-\lambda+1}\|_{Z^{n,\lambda}} \lesssim R^{-1}\|e_j(\nabla^j\langle r\rangle^{-1})\langle r\rangle^{j-\lambda+1}\|_{Z^{n+2,\lambda-2}} \lesssim R^{-1}\|e_j\|_{\ell^1 S(1)}. \tag{4.46}$$

Finally we consider (4.39). Using (4.45) we find

$$\begin{aligned}\|\chi_{>R/2}P^2c_j\nabla^j\langle r\rangle^{-1}\|_{Z^{n,\lambda}} &\lesssim \|c_j\nabla^j\langle r\rangle^{-1}\|_{Z^{n+2,\lambda-\kappa-2}([R/2,\infty))}\\ &\lesssim |c_j|\|\langle r\rangle^{\lambda-\kappa-3-j}\|_{\mathcal{LE}^*([R/2,\infty))}\\ &\lesssim |c_j|\sum_{m>\log\frac{R}{2}}2^{m(\lambda-\kappa-1-j)}.\end{aligned}$$

If $\lambda<\kappa+1+j$, then this yields

$$\|\chi_{>R/2}P^2c_j\nabla^j\langle r\rangle^{-1}\|_{Z^{n,\lambda}}\lesssim R^{-1}|c_j|$$

as desired. Thus the c_0 term fails to be perturbative when $\lambda=\kappa+1$.

Direct calculation easily yields

$$|P^2(c_0\langle r\rangle^{-1})|\lesssim c_0\rho_r^{-\kappa-3}+c_0\rho_\ell^{-\kappa-3}$$

where $\rho_r^{-\kappa-3}\in S_{rad}(r^{-\kappa-3})$ and $\rho_\ell^{-\kappa-3}\in\ell^1S(r^{-\kappa-3})$. Note that $\rho_\ell^{-\kappa-3}\in Z^{\nu,\kappa+1}$ for all ν. We obtain decay as $R\to\infty$ so that

$$\|\chi_{>R/2}c_0\rho_\ell^{-\kappa-3}\|_{Z^{n,\lambda}}\lesssim o_R(1)|c_0|.$$

Thus only the radial term $c_0\rho_r^{-\kappa-3}$, which arises when the radial scalar term in P^2 lands on c_0r^{-1}, fails to be perturbative.

To handle this piece we consider

$$-\Delta w=h\in S_{rad}(r^{-\kappa-3}),\qquad \text{supp } h\subseteq\{r\geq\frac{R}{4}\}.$$

By Lemma 4.2 we have that w can be written as

$$w=c\langle r\rangle^{-1}+e(r)\langle r\rangle^{-\kappa-1}.$$

Using the fact $\partial_r^2(rw)=rh$, we see

$$\|e\|_{S(1)}\lesssim\|h\|_{S_{rad}(r^{-\kappa-3})}.$$

Furthermore, since supp $h \subseteq \{r \geq \frac{R}{4}\}$ we find for $s < \frac{R}{4}$

$$\begin{aligned} |Rw(R)| &= \left| \int_0^R \int_s^\infty \partial_r^2(rw)\ dr\ ds \right| \\ &= \left| \int_0^R \int_s^\infty rh\ dr\ ds \right| \\ &\leq \int_0^R \int_{R/4}^\infty \langle r \rangle^{-\kappa-2}\ dr\ ds \|h\|_{S_{rad}(r^{-\kappa-3})} \\ &\lesssim R^{-\kappa} \|h\|_{S_{rad}(r^{-\kappa-3})}. \end{aligned}$$

Therefore we have

$$|c| = |Rw(R) - e(R)R^{-\kappa}| \lesssim R^{-\kappa} \|h\|_{S_{rad}(r^{-\kappa-3})}.$$

It follows that $w = c\langle r \rangle^{-1} + e(r)\langle r \rangle^{-\kappa-1}$ satisfies the estimate

$$|c| + R^{-\frac{1}{2}} \|e\|_{S(1)} \lesssim R^{-\frac{1}{2}} \|h\|_{S_{rad}(r^{-\kappa-3})}.$$

Using Lemma 4.1 and the above calculations, we have if

$$-\Delta w = h_1 + h_2, \qquad h_1 \in Z^{n,\lambda}, \quad h_2 \in S_{rad}(r^{-q}) \text{ for } q \geq 4$$

where supp $(h_2) \subseteq \{r \geq \frac{R}{4}\}$, then w can be written as in (4.2) with

$$|c_0| + R^{-\frac{1}{2}} \|e_0\|_{S(1)} + \sum_{j=1}^{\lambda-2} |c_j| + \|e_j\|_{\ell^1 S(1)} + \|d(r)\|_{L^\infty} + \|S_r d\|_{\ell^1 S(1)} + \|q\|_{Z^{n+2,\lambda-2}}$$
$$\lesssim \|h_1\|_{Z^{n,\lambda}} + \|h_2\|_{R^{\frac{1}{2}} S_{rad}(r^{-\lambda-2})}.$$

Note that $\ell^1 S(1) \subset S(1)$ so that the change of space for e_0 as compared with Lemma 4.1 causes no problem. We now have (4.36) is perturbative with respect to this estimate since

$$R^{-\frac{1}{2}} \|\chi_{<R/2} c_0 \rho_r^{-\kappa-3}\|_{S_{rad}(r^{-\kappa-3})} \lesssim R^{-\frac{1}{2}} |c_0|.$$

Furthermore, the e_0 terms remain perturbative since our estimate (4.46) came with extra powers of R^{-1} and $\|\chi_{>\frac{R}{2}} e_0\|_{\ell^1 S(1)} \lesssim \log R \|e_0\|_{S(1)}$. This concludes the proof of the proposition. $\square$

Our calculations establishing the error for the estimate $R_\tau g = (R_0 g)e^{-i\tau\langle r\rangle}$ will yield terms of the form $(\partial_r + \frac{1}{r})(\nabla^j\langle r\rangle^{-1})$. The following lemma will be used to handle these terms.

Lemma 4.4. *If $r \geq 2$, then*

$$(\partial_r + \frac{1}{r})(\nabla^j\langle r\rangle^{-1}) = -\frac{1}{2}\Delta\Big(\sum_{k=0}^{j-1}\nabla^{j-1-k}\frac{x}{r}\nabla^k\langle r\rangle^{-1}\Big). \tag{4.47}$$

Proof. In the Appendix we calculate the commutators $[(\partial_r + \frac{1}{r}), \nabla]$ and $[\Delta, \frac{x}{r}]$ and find

$$[(\partial_r + \frac{1}{r}), \nabla] = -\frac{1}{2}[\Delta, \frac{x}{r}] \tag{4.48}$$

(see equation (A.2)).

We use (4.48) to find when $j = 1$

$$\begin{aligned}(\partial_r + \frac{1}{r})\nabla\langle r\rangle^{-1} &= \nabla(\partial_r + \frac{1}{r})r^{-1} + \frac{1}{2}\frac{x}{r}\Delta\langle r\rangle^{-1} - \frac{1}{2}\Delta\Big(\frac{x}{r}r^{-1}\Big)\\ &= -\frac{1}{2}\Delta\Big(\frac{x}{r}\langle r\rangle^{-1}\Big)\end{aligned}$$

for large r. Thus (4.47) holds when $j = 1$.

Now fix J and assume (4.47) holds for $j = J - 1$. We calculate

$$\begin{aligned}(\partial_r + \frac{1}{r})\nabla^J\langle r\rangle^{-1} &= \nabla(\partial_r + \frac{1}{r})\nabla^{J-1}\langle r\rangle^{-1} + \frac{1}{2}\frac{x}{r}\Delta(\nabla^{J-1}\langle r\rangle^{-1}) - \frac{1}{2}\Delta\Big(\frac{x}{r}\nabla^{J-1}\langle r\rangle^{-1}\Big)\\ &= \nabla\Big(-\frac{1}{2}\Delta\sum_{k=0}^{J-2}\nabla^{J-2-k}\frac{x}{r}\nabla^k\langle r\rangle^{-1} + 0 - \frac{1}{2}\Delta\Big(\frac{x}{r}\nabla^{J-1}\langle r\rangle^{-1}\Big)\\ &= -\frac{1}{2}\Delta\Big(\sum_{k=0}^{J-2}\nabla^{J-1-k}\frac{x}{r}\nabla^k\langle r\rangle^{-1} + \frac{x}{r}\nabla^{J-1}\langle r\rangle^{-1}\Big)\\ &= -\frac{1}{2}\Delta\sum_{k=0}^{J-1}\nabla^{J-1-k}\frac{x}{r}\nabla^k\langle r\rangle^{-1}\end{aligned}$$

for $r \geq 2$, as desired. $\square$

We are now ready to calculate the error in the approximation $R_\tau g \approx (R_0 g)e^{-i\tau\langle r\rangle}$. Recall by

(4.1) we have the low frequency part of u is given by

$$u_{<1}(t,x) \cong \int_{\mathbb{R}} \chi_{<1}(|\tau|) R_\tau(-i\tau u_0 + P^1 u_0 - u_1) e^{i\tau t}\, d\tau$$

where $u_0 \in Z^{\nu+1,\kappa}$ and $u_1 \in Z^{\nu,\kappa+1}$. Thus we have

$$u_{<1}(t,x) \cong \int_{\mathbb{R}} \chi_{<1}(|\tau|)\Big(\tau R_\tau(g_\kappa^{\nu+1}) + R_\tau(g_{\kappa+1}^{\nu})\Big) e^{i\tau t}\, d\tau$$

where $g_\kappa^{\nu+1} \in Z^{\nu+1,\kappa}$ and $g_{\kappa+1}^{\nu} \in Z^{\nu,\kappa+1}$. We will establish the error $R_\tau g_\lambda^\nu - (R_0 g_\lambda^\nu) e^{-i\tau\langle r\rangle}$ (which we denote E_λ^ν) for $g_\lambda^\nu \in Z^{\nu,\lambda}$ where $\nu, \lambda \in \mathbb{N}$ with ν sufficiently large and $1 \leq \lambda \leq \kappa + 1$.

Proposition 4.5. *Let $g_\lambda^\nu \in Z^{\nu,\lambda}$ with $1 \leq \lambda \leq \kappa+1$ and $\nu > 3\lambda$. If $|\tau| \lesssim 1$ and $\Im\tau \leq 0$ then*

1. *If $1 \leq \lambda \leq \kappa$ then*

$$R_\tau g_\lambda^\nu = (R_0 g_\lambda^\nu) e^{-i\tau\langle r\rangle} + R_\tau(\chi_{>|\tau|^{-1}} g_\lambda^\nu) + \sum_{m=1}^{\lambda-1} \tau^m (F_m + R_0 \zeta_{\lambda-m}^{\nu-3\lambda}) e^{-i\tau\langle r\rangle} + \tau^\lambda (R_\tau h_{\nu-3\lambda})$$

where $|F_m| \lesssim \langle r\rangle^{-1}$, $\zeta_{\lambda-m}^{\nu-3\lambda} \in Z^{\nu-3\lambda,\lambda-m}$, and $h_{\nu-3\lambda}$ satisfies

$$\sup_{i+j+k+q \leq \nu-3\lambda} \|\langle r\rangle^q (\partial_r + i\tau)^q T^i \Omega^j S^k h\|_{\mathcal{LE}^*} \lesssim 1. \tag{4.49}$$

2. *If $\lambda = \kappa + 1$ then*

$$R_\tau g_{\kappa+1}^\nu = (R_0 g_{\kappa+1}^\nu) e^{-i\tau\langle r\rangle} + R_\tau(\chi_{>|\tau|^{-1}} g_{\kappa+1}^\nu) + \sum_{m=1}^{\kappa} \tau^m (F_m + R_0 \zeta_{\kappa+1-m}^{\nu-3m}) e^{-i\tau r}$$
$$+ \tau^\kappa \epsilon(r,\tau) e^{-i\tau r} + \tau^{\kappa+1}(R_\tau h_{\nu-3\kappa-3})$$

where $|F_m| \lesssim \langle r\rangle^{-1}$, $\zeta_{\kappa+1-m}^{\nu-3m} \in Z^{\nu-3m,\kappa+1-m}$, $\epsilon(r,\tau)$ is of the form

$$\epsilon(r,\tau) = \langle r\rangle^{-1}\epsilon_1(r \wedge |\tau|^{-1}) + \tau\Big(\epsilon_2(r \wedge |\tau|^{-1}) - \epsilon_2(|\tau|^{-1})\Big)$$

with $\epsilon_1, \epsilon_2 \in S(\log r)$, and h satisfies (4.49) *with $\lambda = \kappa + 1$.*

We note that the term $\epsilon(r,\tau)$ in the statement of Proposition 4.5 for $\lambda = \kappa + 1$ arises due to the

$e_0(r)$ term in Proposition 4.3 in the $\lambda = \kappa + 1$ case. The final decay rate is ultimately determined by this $\epsilon(r, \tau)$ term.

Proof. Define E^ν_λ to be the error associated to $R_\tau g^\nu_\lambda$:

$$E^\nu_\lambda := R_\tau g^\nu_\lambda - (R_0 g^\nu_\lambda) e^{-i\tau\langle r\rangle}.$$

We will find an expression for $P_\tau(E^\nu_\lambda)$ and use this to calculate the error. We use ζ^ν_λ to indicate a function in $Z^{\nu,\lambda}$ and allow ζ^ν_λ to change from line to line. The purpose of this notation is to keep track of what function spaces each term of our expression for $P_\tau(E^\nu_\lambda)$ is in while reserving g^ν_λ to indicate the arbitrary but fixed function in $Z^{\nu,\lambda}$ given in the statement of the proposition.

We begin by establishing the following claims:

1. We have the expression

$$\begin{aligned} P_\tau(E^\nu_\lambda) = \chi_{>|\tau|^{-1}}(r) g^\nu_\lambda + \tau\Big(\chi_{<|\tau|^{-1}}(r)\langle r\rangle g^\nu_\lambda \frac{1 - e^{-i\tau\langle r\rangle}}{\tau r}\Big) - \tau^\lambda\Big(\chi_{>|\tau|^{-1}}\tau^{-\lambda} g^\nu_\lambda e^{-i\tau\langle r\rangle}\Big) \\ - 2i\tau(\partial_r + \frac{1}{r})(R_0 g^\nu_\lambda)e^{-i\tau\langle r\rangle} + (\tau\zeta^{\nu-3}_\kappa + \tau^2\zeta^{\nu-2}_{\kappa-1})e^{-i\tau\langle r\rangle}. \end{aligned} \tag{4.50}$$

   Note we then have

$$R_\tau g^\nu_\lambda = (R_0 g^\nu_\lambda)e^{-i\tau\langle r\rangle} + R_\tau(P_\tau(E^\nu_\lambda)). \tag{4.51}$$

2. If $h = \zeta^\nu_0 e^{-i\tau\langle r\rangle}$ then h satisfies

$$\|r^l(\partial_r + i\tau)^l T^i \Omega^j S^k h\|_{\mathcal{LE}^*} \lesssim 1, \quad i + j + k + l \le \nu. \tag{4.52}$$

For Claim 1 (see (4.50)), direct calculation yields for any function ϕ

$$\begin{aligned} P_\tau(\phi e^{-i\tau\langle r\rangle}) = [(\Delta + P^2)\phi]e^{-i\tau\langle r\rangle} - 2i\tau\big[(\partial_r + \frac{1}{r})\phi\big]e^{-i\tau\langle r\rangle} \\ + \Big[\big(\tau(\rho_\ell^{-\kappa-1} + \rho_\ell^{-\kappa}\nabla) + \tau^2\rho_\ell^{-\kappa}\big)\phi\Big]e^{-i\tau\langle r\rangle}. \end{aligned} \tag{4.53}$$

The calculations proving (4.53) can be found in the Appendix (see section A.6.1). Since $P_0 = (\Delta + P^2)$, (4.53) gives

$$P_\tau(E^\nu_\lambda) = P_\tau\left(R_\tau g^\nu_\lambda - (R_0 g^\nu_\lambda)e^{-i\tau\langle r\rangle}\right)$$
$$= g^\nu_\lambda - g^\nu_\lambda e^{-i\tau\langle r\rangle} - 2i\tau\Big[(\partial_r + \frac{1}{r})R_0 g^\nu_\lambda\Big]e^{-i\tau\langle r\rangle} + \Big[\big(\tau(\rho_\ell^{-\kappa-1} + \rho_\ell^{-\kappa}\nabla) + \tau^2\rho_\ell^{-\kappa}\big)R_0 g^\nu_\lambda\Big]e^{-i\tau\langle r\rangle} \tag{4.54}$$

where $\rho^a_\ell \in \ell^1 S(r^a)$ and we allow ρ^a_ℓ to indicate a different function each time it appears.

Using our expansion in Proposition 4.3, we have $R_0 g^\nu_\lambda - q \in S(r^{-1})$ and $q \in Z^{\nu-2,\lambda-2}$. It follows that

$$\rho_\ell^{-\kappa-1} R_0 g^\nu_\lambda \in Z^{\nu-2,\kappa}, \quad \rho_\ell^{-\kappa}\nabla R_0 g^\nu_\lambda \in Z^{\nu-3,\kappa}, \quad \text{and} \quad \rho_\ell^{-\kappa} R_0 g^\nu_\lambda \in Z^{\nu-2,\kappa-1}.$$

Therefore we can write

$$\Big[\big(\tau(\rho_\ell^{-\kappa-1} + \rho_\ell^{-\kappa}\nabla) + \tau^2\rho_\ell^{-\kappa}\big)R_0 g^\nu_\lambda\Big]e^{-i\tau\langle r\rangle} = \tau\zeta^{\nu-3}_\kappa e^{-i\tau\langle r\rangle} + \tau^2\zeta^{\nu-2}_{\kappa-1}e^{-i\tau\langle r\rangle}. \tag{4.55}$$

Furthermore we have

$$\begin{aligned} g^\nu_\lambda - g^\nu_\lambda e^{-i\tau\langle r\rangle} &= \chi_{>|\tau|^{-1}}(r)g^\nu_\lambda + \chi_{<|\tau|^{-1}}(r)g^\nu_\lambda(1 - e^{-i\tau\langle r\rangle}) - \chi_{>|\tau|^{-1}}(r)g^\nu_\lambda e^{-i\tau\langle r\rangle} \\ &= \chi_{>|\tau|^{-1}}(r)g^\nu_\lambda + \tau\chi_{<|\tau|^{-1}}(r)\langle r\rangle g^\nu_\lambda\frac{1 - e^{-i\tau\langle r\rangle}}{\tau\langle r\rangle} - \tau^\lambda\chi_{>|\tau|^{-1}}(r)\tau^{-\lambda}g^\nu_\lambda e^{-i\tau\langle r\rangle}. \end{aligned} \tag{4.56}$$

Combining (4.54), (4.55), and (4.56) then yields (4.50), as desired.

For Claim 2 (see (4.52)), when $r \geq 2$, we find for any $\phi(x)$

$$S^k(\phi e^{-i\tau\langle r\rangle}) = (S^k_r\phi)e^{-i\tau\langle r\rangle}, \qquad \Omega^j(\phi e^{-i\tau\langle r\rangle}) = (\Omega^j\phi)e^{-i\tau\langle r\rangle}$$
$$|T^i(\phi e^{-i\tau\langle r\rangle})| \lesssim \sum_{a=0}^{i}|(T^a\phi)e^{-i\tau\langle r\rangle}|, \qquad (\partial_r + i\tau)^q(\phi e^{-i\tau\langle r\rangle}) = (\partial^q_r\phi)e^{-i\tau\langle r\rangle}.$$

Thus we have

$$\|r^l(\partial_r + i\tau)^l T^i\Omega^j S^k_r(\zeta^\nu_0 e^{-i\tau\langle r\rangle})\|_{\mathcal{LE}^*} \lesssim \|T^i\Omega^j S^{k+l}_r\zeta^\nu_0\|_{\mathcal{LE}^*} \lesssim 1$$

for $i + j + k + l \leq \nu$, as desired. This concludes the proof of the claims above.

We will obtain a recursive formula for E^ν_λ, so we begin by calculating E^ν_1 directly. We note the following, which will help us obtain our recursive formula:

$$R_\tau(g_\lambda^\nu e^{-i\tau\langle r\rangle}) = R_\tau g_\lambda^\nu - R_\tau\Big(g_\lambda^\nu(1-e^{-i\tau\langle r\rangle})\Big). \tag{4.57}$$

Case 1: $\lambda = 1$

We calculate E_1^ν using (4.50) and the expansion for $R_0 g_1^\nu$ given by Proposition 4.3:

$$R_0 g_1^\nu = d(r)\langle r\rangle^{-1} + q \tag{4.58}$$

where $d \in L^\infty$, $S_r d \in \ell^1 S(1)$, and $q \in Z^{\nu-2,\lambda-2}$. The terms

$$\tau^\lambda\Big(\chi_{>|\tau|^{-1}}\tau^{-\lambda} g_\lambda^\nu e^{-i\tau\langle r\rangle}\Big) \quad \text{and} \quad (\tau\zeta_\kappa^{\nu-3} + \tau^2\zeta_{\kappa-1}^{\nu-2})e^{-i\tau\langle r\rangle}$$

in (4.50) are readily seen to be of the form $\tau h_{\nu-3}$ using (4.52). The second term in (4.50) can also be included in $\tau h_{\nu-3}$. The cutoff function restricts this term to the region where $r|\tau| \lesssim 1$, so (4.49) reduces to $h_{\nu-3} \in Z^{\nu-3,0}$. Since $r g_1^\nu \in Z^{\nu,0}$,

$$\partial_r\Big(\frac{1-e^{-i\tau\langle r\rangle}}{\tau r^n}\Big) = \frac{ie^{-i\tau\langle r\rangle}}{r^n} - n\frac{1-e^{-i\tau\langle r\rangle}}{\tau r^{n+1}}, \qquad \partial_r\Big(\frac{ie^{-i\tau\langle r\rangle}}{r^n}\Big) = -n\frac{ie^{-i\tau\langle r\rangle}}{r^{n+1}} + \frac{\tau e^{-i\tau\langle r\rangle}}{r^n}$$

and

$$S\Big(\frac{1-e^{-i\tau\langle r\rangle}}{\tau r}\Big) = 0,$$

we have

$$\langle r\rangle g_1^\nu\frac{1-e^{-i\tau\langle r\rangle}}{\tau\langle r\rangle} \in Z^{\nu,0} \subset Z^{\nu-3,0},$$

as desired. Here and throughout we harmlessly assume $r \geq 2$. We note the above calculations and (4.56) imply

$$R_\tau(g_1^\nu(1-e^{-i\tau\langle r\rangle})) = R_\tau(\chi_{>|\tau|^{-1}} g_1^\nu) + \tau(R_\tau h_\nu). \tag{4.59}$$

This equation will help us handle terms that will arise when $\lambda \geq 2$ by providing a base case for an inductive argument.

For the term $-2i\tau\big[(\partial_r + \frac{1}{r})R_0 g^\nu_\lambda\big]e^{-i\tau\langle r\rangle}$ in (4.50), we use (4.58) and write $\partial_r = r^{-1}S_r$ to find

$$(\partial_r + \frac{1}{r})R_0 g^\nu_1 \in Z^{\nu-3,0},$$

so this term also satisfies (4.49).

Thus we have

$$P_\tau(E^\nu_1) = \chi_{>|\tau|^{-1}} g^\nu_1 + \tau h_{\nu-3}$$

so that

$$E^\nu_1 = R_\tau(\chi_{>|\tau|^{-1}} g^\nu_1) + \tau R_\tau h_{\nu-3} \tag{4.60}$$

as desired.

Case 2: $2 \leq \lambda \leq \kappa + 1$

We again consider each term on the right hand side of (4.50) and use the notation $\zeta^\nu_\lambda \in Z^{\nu,\lambda}$ to track the regularity (indicated by ν) and r decay (indicated by λ) of each term. We note in general $Z^{N_1,L_1} \subset Z^{N_2,L_2}$ for $N_2 \leq N_1$ and $L_2 \leq L_1$. Thus we can replace ζ^ν_λ by $\zeta^{\bar\nu}_{\bar\lambda}$ where $\bar\lambda \leq \lambda$ and $\bar\nu \leq \nu$. The monotonicity of $Z^{\nu,\lambda}$ allows us to collect terms with different regularity and decay and write them as one term:

$$\zeta^{\nu_1}_{\lambda_1} + \zeta^{\nu_2}_{\lambda_2} = \zeta^{\min(\nu_1,\nu_2)}_{\min(\lambda_1,\lambda_2)}. \tag{4.61}$$

For the second term in (4.50), we see

$$\langle r\rangle g^\nu_\lambda \frac{1 - e^{-i\tau\langle r\rangle}}{\tau\langle r\rangle} \in Z^{\nu,\lambda-1}$$

uniformly in τ as $\tau \to 0$. So we can write

$$\tau\chi_{<|\tau|^{-1}}(r)\langle r\rangle g^\nu_\lambda \frac{1 - e^{-i\tau\langle r\rangle}}{\tau\langle r\rangle} = \tau\chi_{<|\tau|^{-1}}(r)\zeta^\nu_{\lambda-1}. \tag{4.62}$$

For the third term in (4.50), we note

$$\chi_{>|\tau|^{-1}}(r)\tau^{-\lambda} g^\nu_\lambda \in Z^{\nu,0}$$

uniformly in τ as $\tau \to 0$. So we can write

$$\tau^\lambda \chi_{>|\tau|^{-1}} \tau^{-\lambda} g_\lambda^\nu e^{-i\tau\langle r\rangle} = \tau^\lambda \zeta_0^\nu e^{-i\tau\langle r\rangle}. \tag{4.63}$$

Substituting (4.62) and (4.63) into (4.50) we find

$$\begin{aligned} P_\tau(E_\lambda^\nu) = \chi_{>|\tau|^{-1}} g_\lambda^\nu + \tau\chi_{<|\tau|^{-1}} \zeta_{\lambda-1}^\nu + \Big(\tau\zeta_\kappa^{\nu-3} + \tau^2\zeta_{\kappa-1}^{\nu-2} + \tau^\lambda\zeta_0^\nu\Big) e^{-i\tau\langle r\rangle} \\ - 2i\tau\big[(\partial_r + \frac{1}{r})R_0 g_\lambda^\nu\big] e^{-i\tau\langle r\rangle}. \end{aligned} \tag{4.64}$$

We claim that terms of the right hand side of (4.64) of the form

$$\tau^m \zeta_{\lambda-m}^\nu e^{-i\tau\langle r\rangle}, \qquad 1 \le m \le \lambda - 1 \tag{4.65}$$

produce error terms which can be handled inductively. To see this, we first consider the case $m = \lambda - 1$, so that $\lambda - m = 1$. In this case (4.57) and (4.59) give

$$R_\tau(g_1^\nu e^{-i\tau\langle r\rangle}) = R_\tau(\chi_{<|\tau|^{-1}} g_1^\nu) + \tau(R_\tau h_\nu)).$$

Therefore after applying R_τ to both sides of (4.64), we see terms on the right hand side of the form (4.65) with $m = \lambda - 1$ become

$$\tau^{\lambda-1} R_\tau(\zeta_1^\nu e^{-i\tau\langle r\rangle}) = \tau^{\lambda-1} R_\tau(\chi_{<|\tau|^{-1}} \zeta_1^\nu) + \tau^\lambda(R_\tau h_\nu), \tag{4.66}$$

and we can appeal to case 1 of the proposition to handle the first term, while the second term is expected to appear in E_λ^ν (In fact this term has more regularity than the h term in the statement of the proposition. We will see the last term on the right hand side of (4.64) limits the amount of regularity we can get for h in our error.)

Now consider (4.65) for $1 \le m \le \lambda - 2$. Substituting (4.62) and (4.63) into (4.56) gives

$$R_\tau(g_\lambda^\nu(1 - e^{-i\tau\langle r\rangle})) = R_\tau(\chi_{>|\tau|^{-1}} g_\lambda^\nu) + \tau R_\tau(\chi_{<|\tau|^{-1}} \zeta_{\lambda-1}^\nu)) + \tau^\lambda(R_\tau h_\nu). \tag{4.67}$$

Then (4.57) and (4.67) give

$$R_\tau(g^\nu_\lambda e^{-i\tau\langle r\rangle}) = R_\tau(\chi_{<|\tau|^{-1}} g^\nu_\lambda) + \tau R_\tau(\chi_{<|\tau|^{-1}}\zeta^\nu_{\lambda-1}) + \tau^\lambda(R_\tau h_\nu).$$

Therefore after applying R_τ to both sides of (4.64), we see terms on the right hand side that are of the form (4.65) with $1 \le m \le \lambda - 2$ become

$$\tau^m R_\tau(\zeta^\nu_{\lambda-m} e^{-i\tau\langle r\rangle}) = \tau^m R_\tau(\chi_{<|\tau|^{-1}}\zeta^\nu_{\lambda-m}) + \tau^{m+1} R_\tau(\chi_{<|\tau|^{-1}}\zeta^\nu_{\lambda-m-1}) + \tau^\lambda(R_\tau h_\nu), \tag{4.68}$$

and we can proceed inductively for the first 2 terms, while the last term is expected to appear in E^ν_λ.

Next we consider the term $-2i\tau\big[(\partial_r + \frac{1}{r})R_0 g^\nu_\lambda\big]e^{-i\tau\langle r\rangle}$ in (4.64). By Proposition 4.3 we have

$$R_0 g^\nu_\lambda = \sum_{j=0}^{\lambda-2}\Big(c_j\nabla^j\langle r\rangle^{-1} + e_j(r)(\nabla^j\langle r\rangle^{-1})\langle r\rangle^{j-\lambda+1}\Big) + d(r)\nabla^{\lambda-1}\langle r\rangle^{-1} + q(x)$$

where

$$|c_0| + \sum_{j=1}^{\lambda-2}(|c_j| + \|e_j\|_{\ell^1 S(1)}) + \|d\|_{L^\infty} + \|S_r d\|_{\ell^1 S(1)} + \|q\|_{Z^{\nu-2,\lambda-2}} \lesssim 1$$

for all $2 \le \lambda \le \kappa + 1$ and

$$\|e_0\|_{\ell^1 S(1)} \lesssim 1 \quad \text{if } \lambda \le \kappa$$

$$\|e_0\|_{S(1)} \lesssim 1 \quad \text{if } \lambda = \kappa + 1.$$

We note $(\partial_r + \frac{1}{r})\langle r\rangle^{-1} = 0$ so the term $2i\tau(\partial_r + \frac{1}{r})c_0\langle r\rangle^{-1}$ vanishes. It is left to consider the terms

(A) $-2i\tau\big[(\partial_r + \frac{1}{r})c_j\nabla^j\langle r\rangle^{-1}\big]e^{-i\tau\langle r\rangle}$ for $1 \le j \le \lambda - 2$

(B) $-2i\tau(\partial_r + \frac{1}{r})\big[e_j(r)(\nabla^j\langle r\rangle^{-1})\langle r\rangle^{j-\lambda+1}\big]e^{-i\tau\langle r\rangle}$ for $1 \le j \le \lambda - 2$

(C) $-2i\tau\big[(\partial_r + \frac{1}{r})e_0\langle r\rangle^{-\lambda}\big]e^{-i\tau\langle r\rangle}$ (here the cases $\lambda \le \kappa$ and $\lambda = \kappa + 1$ must be considered separately)

(D) $-2i\tau\big[(\partial_r + \frac{1}{r})d(r)\nabla^{\lambda-1}\langle r\rangle^{-1}\big]e^{-i\tau\langle r\rangle}$

(E) $-2i\tau\big[(\partial_r + \frac{1}{r})q(x)\big]e^{-i\tau\langle r\rangle}$.

We will handle term C last since only this term requires different arguments depending on the value of λ. For the other terms, we proceed in order of difficulty, starting with the simplest.

For term E we have $q \in Z^{\nu-2,\lambda-2}$, so writing $\partial_r = r^{-1}S_r$ we see $(\partial_r + \frac{1}{r})q \in Z^{\nu-3,\lambda-1}$. Thus we can write

$$-2i\tau\big[(\partial_r + \frac{1}{r})q(x)\big]e^{-i\tau\langle r\rangle} = \tau\zeta^{\nu-3}_{\lambda-1}e^{-i\tau\langle r\rangle}. \tag{4.69}$$

For term B we have $e_j \in \ell^1 S(1)$ so

$$e_j(r)(\nabla^j\langle r\rangle^{-1})\langle r\rangle^{j-\lambda+1} \in \ell^1 S(r^{-\lambda})$$

and

$$(\partial_r + \frac{1}{r})e_j(\nabla^j\langle r\rangle^{-1})\langle r\rangle^{j-\lambda+1} \in \ell^1 S(r^{-\lambda-1}).$$

Since $\ell^1 S(r^{-2}) \subseteq Z^{N,0}$ for any N, we can write

$$-2i\tau\big[(\partial_r + \frac{1}{r})e_j(r)(\nabla^j\langle r\rangle^{-1})\langle r\rangle^{j-\lambda+1}\big]e^{-i\tau\langle r\rangle} = \tau\zeta^{\nu-3}_{\lambda-1}e^{-i\tau\langle r\rangle}. \tag{4.70}$$

Terms A, C, and D require more work. Our argument for these terms will use the following calculations. Let

$$-2i\tau\big[(\partial_r + \frac{1}{r})\varphi_0\big]e^{-i\tau\langle r\rangle}$$

indicate a term that arises in our expression for $P_\tau(E^\nu_\lambda)$ in (4.50). If φ_1 satisfies

$$-\frac{1}{2}\Delta\varphi_1 = (\partial_r + \frac{1}{r})\varphi_0$$

then by (4.53), we have

$$\begin{aligned}-2i\tau(\partial_r + \frac{1}{r})(\varphi_0)e^{-i\tau\langle r\rangle} &= P_\tau(\tau i\varphi_1 e^{-i\tau\langle r\rangle}) - 2\tau^2\big[(\partial_r + \frac{1}{r})\varphi_1\big]e^{-i\tau\langle r\rangle}\\ &\quad + \tau\Big[\Big(\tau(\rho_\ell^{-\kappa-1} + \rho_\ell^{-\kappa}\nabla) + \tau^2\rho_\ell^{-\kappa}\Big)\varphi_1\Big]e^{-i\tau\langle r\rangle} - i\tau(P^2\varphi_1)e^{-i\tau\langle r\rangle}.\end{aligned} \tag{4.71}$$

Furthermore, if $\varphi_1 \in S(r^{-1})$, then we can write

$$\tau\Big[\Big(\tau(\rho_\ell^{-\kappa-1} + \rho_\ell^{-\kappa}\nabla) + \tau^2\rho_\ell^{-\kappa}\Big)\varphi_1\Big]e^{-i\tau\langle r\rangle} - i\tau(P^2\varphi_1)e^{-i\tau\langle r\rangle} = \Big(\tau\zeta^\nu_\kappa + \tau^2\zeta^\nu_\kappa + \tau^3\zeta^\nu_{\kappa-1}\Big)e^{-i\tau\langle r\rangle}$$

where the term $\tau\zeta^\nu_\kappa$ is limited in its decay rate by the P^2 term which is in $S_{rad}(r^{-\kappa-2})$, and we write (4.71) as

$$\begin{aligned}-2i\tau(\partial_r+\frac{1}{r})(\varphi_0)e^{-i\tau\langle r\rangle} = i\tau P_\tau(\varphi_1 e^{-i\tau\langle r\rangle}) - 2\tau^2\big[(\partial_r+\frac{1}{r})\varphi_1\big]e^{-i\tau\langle r\rangle} \\ + \Big(\tau\zeta^\nu_\kappa+\tau^2\zeta^\nu_\kappa+\tau^3\zeta^\nu_{\kappa-1}\Big)e^{-i\tau\langle r\rangle}.\end{aligned} \tag{4.72}$$

Repeating the above argument this time for $2i\tau^2(\partial_r+\frac{1}{r})(i\varphi_1)e^{-i\tau\langle r\rangle}$ with φ_2 satisfying

$$-\frac{1}{2}\Delta\varphi_2 = (\partial_r+\frac{1}{r})\varphi_1$$

and plugging the resulting expression for $2\tau^2(\partial_r+\frac{1}{r})(\varphi_1)e^{-i\tau\langle r\rangle}$ into (4.72) we find

$$\begin{aligned}-2i\tau(\partial_r+\frac{1}{r})(\varphi_0)e^{-i\tau\langle r\rangle} = P_\tau\Big(\tau(i\varphi_1)e^{-i\tau\langle r\rangle}+\tau^2(-i^2\varphi_2)e^{-i\tau\langle r\rangle}\Big) + 2i\tau^3\big[(\partial_r+\frac{1}{r})\varphi_2\big]e^{-i\tau\langle r\rangle} \\ + \Big(\tau\zeta^\nu_\kappa+\tau^2\zeta^\nu_\kappa+\tau^3(\zeta^\nu_\kappa+\zeta^\nu_{\kappa-1})+\tau^4\zeta^\nu_\kappa\Big)e^{-i\tau\langle r\rangle}\end{aligned}$$

as long as $\varphi_1,\varphi_2\in S(r^{-1})$. Repeating this process with φ_n satisfying

$$-\frac{1}{2}\Delta\varphi_n = (\partial_r+\frac{1}{r})\varphi_{n-1}$$

we have

$$\begin{aligned}-2i\tau(\partial_r+\frac{1}{r})(\varphi_0)e^{-i\tau\langle r\rangle} = \sum_{a=1}^{n}\Big(-P_\tau((-i\tau)^a\varphi_a e^{-i\tau\langle r\rangle}) + (\tau^a\zeta^\nu_\kappa+\tau^{a+1}\zeta^\nu_\kappa+\tau^{a+2}\zeta^\nu_{\kappa-1})e^{-i\tau\langle r\rangle}\Big) \\ -2(-i\tau)^{n+1}(\partial_r+\frac{1}{r})(\varphi_n)e^{-i\tau\langle r\rangle}\end{aligned} \tag{4.73}$$

as long as $\varphi_a\in S(r^{-1})$ for $a\le n$.

To handle terms A and D, we first obtain an expression for

$$-2i\tau(\partial_r+\frac{1}{r})(\nabla^j\langle r\rangle^{-1})e^{-i\tau\langle r\rangle}$$

using Lemma 4.4. Setting $\varphi_0^j = \nabla^j \langle r\rangle^{-1}$ we see (4.47) can be written as

$$(\partial_r + \frac{1}{r})\varphi_0^j = -\frac{1}{2}\Delta\Big(\sum_{k=0}^{j-1} \nabla^{j-1-k}\frac{x}{r}\varphi_0^k\Big) \tag{4.74}$$

so that setting

$$\varphi_1^j := \sum_{k=0}^{j-1} \nabla^{j-1-k}\frac{x}{r}\varphi_0^k \tag{4.75}$$

yields

$$-\frac{1}{2}\Delta\varphi_1^j = (\partial_r + \frac{1}{r})\varphi_0^j.$$

In order to use the argument obtaining (4.73), we then need to find φ_2^j such that

$$-\frac{1}{2}\Delta\varphi_2^j = (\partial_r + \frac{1}{r})\varphi_1^j.$$

We claim that in general, if a family of functions φ^j satisfies

$$(\partial_r + \frac{1}{r})\varphi^j = -\frac{1}{2}\Delta\Big(\sum_{k=a}^{j-1} \nabla^{j-1-k}\frac{x}{r}\varphi^k\Big) \tag{4.76}$$

and $(\partial_r + \frac{1}{r})\varphi^a = 0$, then

$$(\partial_r + \frac{1}{r})\sum_{k=a}^{j-1} \nabla^{j-1-k}\frac{x}{r}\varphi^k = -\frac{1}{2}\Delta\sum_{k=a+1}^{j-1} \nabla^{j-1-k}\frac{x}{r}\Big(\sum_{\ell=a}^{k-1}\nabla^{k-1-\ell}\frac{x}{r}\varphi^\ell\Big). \tag{4.77}$$

The claim is proved by direct calculation in Lemma 4.6 below. We see from (4.74) that φ_0^j satisfies (4.76) with $a = 0$. Furthermore $(\partial_r + \frac{1}{r})\varphi_0^0 = 0$. Then using our definition for φ_1^j in (4.75) we have

$$(\partial_r + \frac{1}{r})\varphi_1^j = -\frac{1}{2}\Delta\sum_{k=1}^{j-1} \nabla^{j-1-k}\frac{x}{r}\varphi_1^k =: -\frac{1}{2}\Delta\varphi_2^j$$

by (4.77). Now φ_1^j satisfies (4.76) with $a = 1$ and

$$(\partial_r + \frac{1}{r})\varphi_1^1 = (\partial_r + \frac{1}{r})\frac{x}{r}\langle r\rangle^{-1} = 0$$

so we can iterate the process again. We define

$$\varphi_\ell^j := \sum_{k=\ell-1}^{j-1} \nabla^{j-1-k} \frac{x}{r} \varphi_{\ell-1}^k \tag{4.78}$$

and see $(\partial_r + \frac{1}{r})\varphi_j^j = (\partial_r + \frac{1}{r})\big(\frac{x}{r}\big)^j \langle r\rangle^{-1} = 0$ so the assumptions of the claim are satisfied at each iteration and we find

$$\begin{aligned}
(\partial_r + \frac{1}{r})\varphi_0^j &= -\frac{1}{2}\Delta \sum_{k=0}^{j-1} \nabla^{j-1-k} \frac{x}{r}\varphi_0^k =: -\frac{1}{2}\Delta\varphi_1^j \\
(\partial_r + \frac{1}{r})\varphi_1^j &= -\frac{1}{2}\Delta \sum_{k=1}^{j-1} \nabla^{j-1-k} \frac{x}{r}\varphi_1^k =: -\frac{1}{2}\Delta\varphi_2^j \\
&\vdots \\
(\partial_r + \frac{1}{r})\varphi_n^j &= -\frac{1}{2}\Delta \sum_{k=n}^{j-1} \nabla^{j-1-k} \frac{x}{r}\varphi_n^k =: -\frac{1}{2}\Delta\varphi_{n+1}^j.
\end{aligned}$$

We note $\varphi_n^j \in S(r^{-j-1+n})$ so $\varphi_n^j \in S(r^{-1})$ for $n \leq j$. Now by (4.73) we have

$$\begin{aligned}
&-2i\tau(\partial_r + \frac{1}{r})(\nabla^j\langle r\rangle^{-1})e^{-i\tau\langle r\rangle} \\
&\qquad = \sum_{a=1}^{j} \Big(- P_\tau((-i\tau)^a \varphi_a^j e^{-i\tau\langle r\rangle}) + (\tau^a\zeta_\kappa^\nu + \tau^{a+1}\zeta_\kappa^\nu + \tau^{a+2}\zeta_{\kappa-1}^\nu)e^{-i\tau\langle r\rangle}\Big)
\end{aligned} \tag{4.79}$$

since $(\partial_r + \frac{1}{r})\varphi_j^j = 0$.

Now to handle term A, we multiply both sides of (4.79) by the constant factor c_j to find

$$\begin{aligned}
&-2i\tau(\partial_r + \frac{1}{r})(c_j\nabla^j\langle r\rangle^{-1})e^{-i\tau\langle r\rangle} \\
&\qquad = \sum_{a=1}^{j} \Big(- P_\tau((-i\tau)^a c_j \varphi_a^j e^{-i\tau\langle r\rangle}) + c_j(\tau^a\zeta_\kappa^\nu + \tau^{a+1}\zeta_\kappa^\nu + \tau^{a+2}\zeta_{\kappa-1}^\nu)e^{-i\tau\langle r\rangle}\Big) \\
&\qquad = \sum_{a=1}^{j} \Big(P_\tau(\tau^a F_a e^{-i\tau\langle r\rangle}) + (\tau^a\zeta_\kappa^\nu + \tau^{a+1}\zeta_\kappa^\nu + \tau^{a+2}\zeta_{\kappa-1}^\nu)e^{-i\tau\langle r\rangle}\Big)
\end{aligned} \tag{4.80}$$

where $|F_a| \lesssim \langle r\rangle^{-1}$. In the last line of (4.80), we absorbed the constant into the functions ζ_κ^ν and $\zeta_{\kappa-1}^\nu$, since these are allowed to change from line to line.

Next we handle term D. Using (4.79) and the fact that $S_r d \in \ell^1 S(1)$, we calculate

$$\begin{aligned}
&-2i\tau(\partial_r + \frac{1}{r})(d(r)\nabla^{\lambda-1}\langle r\rangle^{-1})e^{-i\tau\langle r\rangle} \\
&\quad = -2i\tau(\partial_r d)(\nabla^{\lambda-1}\langle r\rangle^{-1})e^{-i\tau\langle r\rangle} - d(r)2i\tau(\partial_r + \frac{1}{r})(\nabla^{\lambda-1}\langle r\rangle^{-1}) \\
&\quad = \tau\zeta^{\nu}_{\lambda-1}e^{-i\tau\langle r\rangle} + \sum_{a=1}^{\lambda-1} d(r)P_\tau((-i\tau)^a\varphi_a^{\lambda-1}e^{-i\tau\langle r\rangle}) + (\tau^a\zeta^\nu_\kappa + \tau^{a+1}\zeta^\nu_\kappa + \tau^{a+2}\zeta^\nu_{\kappa-1})e^{-i\tau\langle r\rangle} \qquad (4.81) \\
&\quad = \tau\zeta^{\nu}_{\lambda-1}e^{-i\tau\langle r\rangle} + \sum_{a=1}^{\lambda-1} P_\tau(d(r)(-i\tau)^a\varphi_a^{\lambda-1}e^{-i\tau\langle r\rangle}) - [P_\tau, d(r)](-i\tau)^a\varphi_a^{\lambda-1}e^{-i\tau\langle r\rangle} \\
&\qquad + \left(\tau^a\zeta^\nu_\kappa + \tau^{a+1}\zeta^\nu_\kappa + \tau^{a+2}\zeta^\nu_{\kappa-1}\right)e^{-i\tau\langle r\rangle}.
\end{aligned}$$

We find by direct calculation

$$[P_\tau, d]\varphi_a^{\lambda-1}e^{-i\tau\langle r\rangle} = \left(\rho_\ell^{-\lambda+a-2} + \tau\rho_\ell^{-\lambda+a-1}\right)e^{-i\tau\langle r\rangle} \qquad (4.82)$$

where $\rho_\ell^{-\lambda+a-2} \in \ell^1 S(r^{-\lambda+a-2})$ and $\rho_\ell^{-\lambda+a-1} \in \ell^1 S(r^{-\lambda+a-1})$. Since $\ell^1 S(r^{-2}) \subset Z^{N,0}$ for any N, (4.81) and (4.82) yield

$$\begin{aligned}
&-2i\tau(\partial_r + \frac{1}{r})(d(r)\nabla^{j}\langle r\rangle^{-1})e^{-i\tau\langle r\rangle} \\
&\quad = \sum_{a=1}^{j} P_\tau\left(\tau^a F_a e^{-i\tau\langle r\rangle}\right) + (\tau^a(\zeta^\nu_\kappa + \zeta^\nu_{\lambda-a}) + \tau^{a+1}(\zeta^\nu_\kappa + \zeta^\nu_{\lambda-a-1}) + \tau^{a+2}\zeta^\nu_{\kappa-1})e^{-i\tau\langle r\rangle}.
\end{aligned} \qquad (4.83)$$

We now have expressions for terms A, B, D, and E. Before handling term C we recap the current status of our expression for $P_\tau(E^\nu_\lambda)$. Substituting (4.69), (4.70), (4.80), and (4.83) into (4.64) and simplifying using the fact that $Z^{N_1,M_1} \subset Z^{N_2,M_2}$ for $N_2 \le N_1$ and $M_2 \le M_1$ yields

$$\begin{aligned}
P_\tau(E^\nu_\lambda) = {} & \chi_{>|\tau|^{-1}}g^\nu_\lambda + \sum_{a=1}^{\lambda-1}\left(P_\tau(\tau^a F_a e^{-i\tau\langle r\rangle})\right) + \tau\chi_{<|\tau|^{-1}}\zeta^\nu_{\lambda-1} + \left(\tau\zeta^{\nu-3}_{\lambda-1} + \tau^2\zeta^{\nu-2}_{\lambda-2}\right)e^{-i\tau\langle r\rangle} \\
& + \sum_{m=3}^{\lambda}\tau^m\zeta^\nu_{\lambda-m}e^{-i\tau\langle r\rangle} + 2\tau(\partial_r + \frac{1}{r})(e_0\langle r\rangle^{-\lambda})e^{-i\tau\langle r\rangle}
\end{aligned} \qquad (4.84)$$

for $1 \le \lambda \le \kappa + 1$. For term C we consider the cases $2 \le \lambda \le \kappa$ and $\lambda = \kappa + 1$ separately.

Case 2(a): $2 \le \lambda \le \kappa$

Here we have $e_0 \in \ell^1 S(1)$ so $e_0\langle r\rangle^{-\lambda} \in \ell^1 S(r^{-\lambda})$ and we find

$$2\tau(\partial_r + \frac{1}{r})(e_0(r)\langle r\rangle^{-\lambda})e^{-i\tau\langle r\rangle} = \tau\zeta^{\nu}_{\lambda-1}e^{-i\tau\langle r\rangle}. \tag{4.85}$$

A term of this form already appears in (4.84) (since $Z^{\nu,\lambda-1} \subset Z^{\nu-3,\lambda-1}$), so we have

$$\begin{aligned} P_\tau(E^\nu_\lambda) = \chi_{>|\tau|^{-1}} g^\nu_\lambda + \sum_{a=1}^{\lambda-1}\Big(P_\tau(\tau^a F_a e^{-i\tau\langle r\rangle})\Big) + \tau\chi_{<|\tau|^{-1}}\zeta^\nu_{\lambda-1} + \Big(\tau\zeta^{\nu-3}_{\lambda-1} + \tau^2\zeta^{\nu-2}_{\lambda-2}\Big)e^{-i\tau\langle r\rangle} \\ + \sum_{m=3}^{\lambda}\tau^m\zeta^\nu_{\lambda-m}e^{-i\tau\langle r\rangle}. \end{aligned} \tag{4.86}$$

Applying R_τ to both sides of (4.86) we find

$$\begin{aligned} E^\nu_\lambda = R_\tau(\chi_{>|\tau|^{-1}} g^\nu_\lambda) + \sum_{a=1}^{\lambda-1}(\tau^a F_a e^{-i\tau\langle r\rangle}) + \tau R_\tau(\chi_{<|\tau|^{-1}}\zeta^\nu_{\lambda-1}) + \tau R_\tau(\zeta^{\nu-3}_{\lambda-1}e^{-i\tau\langle r\rangle}) \\ + \tau^2 R_\tau(\zeta^{\nu-2}_{\lambda-2}e^{-i\tau\langle r\rangle}) + \sum_{m=3}^{\lambda-1}\tau^m R_\tau(\zeta^\nu_{\lambda-m}e^{-i\tau\langle r\rangle}) + \tau^\lambda(R_\tau h_\nu). \end{aligned} \tag{4.87}$$

Then by (4.66) and (4.68) we have

$$\begin{aligned} E^\nu_\lambda = R_\tau(\chi_{>|\tau|^{-1}} g^\nu_\lambda) + \sum_{a=1}^{\lambda-1}(\tau^a F_a e^{-i\tau\langle r\rangle}) + \tau R_\tau(\chi_{<|\tau|^{-1}}\zeta^{\nu-3}_{\lambda-1}) + \tau^2 R_\tau(\chi_{<|\tau|^{-1}}\zeta^{\nu-3}_{\lambda-2}) \\ + \sum_{m=3}^{\lambda-1}\tau^m R_\tau(\chi_{<|\tau|^{-1}}\zeta^\nu_{\lambda-m}) + \tau^\lambda(R_\tau h_{\nu-3}). \end{aligned} \tag{4.88}$$

Part 1 of the proposition then follows by induction in λ and the established base case for $\lambda = 1$. We note the term $\tau R_\tau(\chi_{<|\tau|^{-1}}\zeta^{\nu-3}_{\lambda-1})$ leads to the loss of regularity for h.

Case 2(b): $\lambda = \kappa + 1$

We proceed as we did for terms A and D. Set

$$\varphi_0 := e_0\langle r\rangle^{-\kappa-1} \in S_{rad}(r^{-\kappa-1}).$$

Here we have the advantage that $(\partial_r + \frac{1}{r})e_0\langle r\rangle^{-\kappa-1}$ is radial and we use Lemma 4.2 to find

$$
\begin{aligned}
(\partial_r + \frac{1}{r})\varphi_0 &= -\frac{1}{2}\Delta\varphi_1, \qquad \varphi_1 \in S_{rad}(r^{-\kappa}) \\
(\partial_r + \frac{1}{r})\varphi_1 &= -\frac{1}{2}\Delta\varphi_2, \qquad \varphi_2 \in S_{rad}(r^{-\kappa+1}) \\
&\vdots \\
(\partial_r + \frac{1}{r})\varphi_{\kappa-2} &= -\frac{1}{2}\Delta\varphi_{\kappa-1}, \qquad \varphi_{\kappa-1} \in S_{rad}(r^{-2}).
\end{aligned}
$$

Then by (4.73) we have

$$
\begin{aligned}
&-2i\tau(\partial_r + \frac{1}{r})(e_0(r)\langle r\rangle^{-\kappa-1})e^{-i\tau\langle r\rangle} \\
&= \sum_{a=1}^{\kappa-1}\Big(- P_\tau((-i\tau)^a F_a e^{-i\tau\langle r\rangle}) + (\tau^a\zeta_\kappa^\nu + \tau^{a+1}\zeta_\kappa^\nu + \tau^{a+1}\zeta_{\kappa-1}^\nu)e^{-i\tau\langle r\rangle}\Big) \\
&\quad - 2(-i\tau)^\kappa(\partial_r + \frac{1}{r})(\varphi_{\kappa-1})e^{-i\tau\langle r\rangle}.
\end{aligned} \tag{4.89}
$$

Since $\varphi_{\kappa-1} \in S_{rad}(r^{-2})$ we have

$$
(\partial_r + \frac{1}{r})\varphi_{\kappa-1} \in S_{rad}(r^{-3}).
$$

By Lemma 4.2 there exists an $\epsilon_1 \in S_{rad}(\log r)$ such that

$$
-\frac{1}{2}\Delta\langle r\rangle^{-1}\epsilon_1(r) = (\partial_r + \frac{1}{r})\varphi_{\kappa-1}.
$$

We wish to avoid the logarithmic growth in r, so we use the modified function

$$
\tilde{\varphi}_\kappa := \langle r\rangle^{-1}\epsilon_1(r \wedge |\tau|^{-1}).
$$

Now we have

$$
\begin{aligned}
-\frac{1}{2}\Delta\tilde{\varphi}_\kappa &= \chi_{<|\tau|^{-1}}(\partial_r + \frac{1}{r})\varphi_{\kappa-1} \\
&= (\partial_r + \frac{1}{r})\varphi_{\kappa-1} - \chi_{>|\tau|^{-1}}(\partial_r + \frac{1}{r})\varphi_{\kappa-1}.
\end{aligned}
$$

Using (4.53) we find

$$
\begin{aligned}
&2\tau^{\kappa}(\partial_r + \frac{1}{r})\varphi_{\kappa-1}e^{-i\tau\langle r\rangle} \\
&\qquad = P_\tau\Big(\tau^{\kappa}(-\tilde{\varphi}_\kappa)e^{-i\tau\langle r\rangle}\Big) + 2i\tau^{\kappa+1}(\partial_r + \frac{1}{r})(\tilde{\varphi}_\kappa)e^{-i\tau\langle r\rangle} \\
&\qquad\quad + \tau^{\kappa}\Big(\tau(\rho_\ell^{-\kappa-1} + \rho_\ell^{-\kappa}\nabla) + \tau^2\rho_\ell^{-\kappa}\Big)(\tilde{\varphi}_\kappa)e^{-i\tau\langle r\rangle} + \tau^{\kappa}P^2(\tilde{\varphi}_\kappa)e^{-i\tau\langle r\rangle} \\
&\qquad\quad + 2\tau^{\kappa}\chi_{>|\tau|^{-1}}(\partial_r + \frac{1}{r})\varphi_{\kappa-1}e^{-i\tau\langle r\rangle} \\
&\qquad = P_\tau(\tau^{\kappa}(-\tilde{\varphi}_\kappa)e^{-i\tau\langle r\rangle}) + 2\tau^{\kappa+1}\chi_{<|\tau|^{-1}}\langle r\rangle^{-1}(\partial_r\epsilon_1(r))e^{-i\tau\langle r\rangle} \\
&\qquad\quad + \Big(\tau^{\kappa}\zeta_\kappa^{\nu} + \tau^{\kappa+1}(\zeta_{\kappa-1}^{\nu} + \zeta_0^{\nu}) + \tau^{\kappa+2}\zeta_{\kappa-2}^{\nu}\Big)e^{-i\tau\langle r\rangle}.
\end{aligned} \tag{4.90}
$$

Here we used the fact

$$|\tau|^{-1}\chi_{>|\tau|^{-1}}(\partial_r + \frac{1}{r})\varphi_{\kappa-1} \in \ell^1 S(r^{-2})$$

because $(\partial_r + \frac{1}{r})\varphi_{\kappa-1} \in \ell^1 S(r^{-3})$, and the cutoff function allows us to pull out a τ factor when summing in the $\ell^1 S(r^{-2})$ norm.

We still need to handle the term

$$2\tau^{\kappa+1}\chi_{<|\tau|^{-1}}\langle r\rangle^{-1}(\partial_r\epsilon_1(r))e^{-i\tau\langle r\rangle}.$$

We have $\langle r\rangle^{-1}(\partial_r\epsilon_1(r)) \in S_{rad}(r^{-2})$, so by Lemma 4.2, there exists an $\epsilon_2 \in S_{rad}(\log r)$ such that

$$-\frac{1}{2}\Delta\epsilon_2 = \langle r\rangle^{-1}(\partial_r\epsilon_1(r)) \in S_{rad}(r^{-2}).$$

To remove the logarithmic growth in r we use the modified function

$$\tilde{\varphi}_{\kappa+1} := \epsilon_2(r \wedge |\tau|^{-1}) - \epsilon_2(|\tau|^{-1})$$

and find

$$-\frac{1}{2}\Delta\tilde{\varphi}_{\kappa+1} = \chi_{<|\tau|^{-1}}\langle r\rangle^{-1}(\partial_r\epsilon_1(r)).$$

Then by (4.53) we have

$$
\begin{aligned}
&2\tau^{\kappa+1}\chi_{<|\tau|^{-1}}\langle r\rangle^{-1}(\partial_r\epsilon_1(r))e^{-i\tau\langle r\rangle}\\
&\qquad= P_\tau(\tau^{\kappa+1}(-\tilde{\varphi}_{\kappa+1})e^{-i\tau\langle r\rangle}) + 2\tau^{\kappa+2}(\partial_r+\frac{1}{r})(\tilde{\varphi}_{\kappa+1})e^{-i\tau\langle r\rangle}\\
&\qquad\quad + \tau^{\kappa+1}\Big(\tau(\rho_\ell^{-\kappa-1}+\rho_\ell^{-\kappa}\nabla)+\tau^2\rho_\ell^{-\kappa}\Big)(\tilde{\varphi}_{\kappa+1})e^{-i\tau\langle r\rangle} + (P^2\tilde{\varphi}_{\kappa+1})e^{-i\tau\langle r\rangle}\\
&\qquad= P_\tau(\tau^{\kappa+1}(-\tilde{\varphi}_{\kappa+1})e^{-i\tau\langle r\rangle}) + \tau^{\kappa+1}\zeta_0^\nu e^{-i\tau\langle r\rangle} + \Big(\tau^\kappa\zeta_{\kappa-1}^\nu + \tau^{\kappa+1}\zeta_{\kappa-1}^\nu\Big)e^{-i\tau\langle r\rangle}.
\end{aligned}
\tag{4.91}
$$

Combining (4.90), (4.91), and (4.89) we find

$$
\begin{aligned}
&2\tau(\partial_r+\frac{1}{r})(e_0(r)\langle r\rangle^{-\kappa-1})e^{-i\tau\langle r\rangle}\\
&= \sum_{a=1}^{\kappa-1}\Big(P_\tau(\tau^a F_a e^{-i\tau\langle r\rangle}) + (\tau^a\zeta_\kappa^\nu + \tau^{a+1}\zeta_\kappa^\nu + \tau^{a+1}\zeta_{\kappa-1}^\nu)e^{-i\tau\langle r\rangle}\Big) + (\tau^\kappa\zeta_1^\nu + \tau^{\kappa+1}\zeta_0^\nu)e^{-i\tau\langle r\rangle}\\
&\quad - \tau^\kappa P_\tau\Big((\langle r\rangle^{-1}\epsilon_1(r\wedge|\tau|^{-1}) + \tau(\epsilon_2(r\wedge|\tau|^{-1}-\epsilon_2(|\tau|^{-1}))e^{-i\tau\langle r\rangle}\Big).
\end{aligned}
\tag{4.92}
$$

Now combining (4.84) and (4.92) yields

$$
\begin{aligned}
P_\tau(E_{\kappa+1}^\nu) &= \chi_{>|\tau|^{-1}}g_\lambda^\nu + \sum_{a=1}^{\kappa}\Big(P_\tau(\tau^a F_a e^{-i\tau\langle r\rangle})\Big) + \tau\chi_{<|\tau|^{-1}}\zeta_\kappa^\nu + \tau\zeta_\kappa^{\nu-3}e^{-i\tau\langle r\rangle}\\
&\quad + \tau^2\zeta_{\kappa-1}^{\nu-2}e^{-i\tau\langle r\rangle} + \sum_{m=2}^{\kappa+1}\tau^m\zeta_{\kappa+1-m}^\nu e^{-i\tau\langle r\rangle} - \tau^\kappa P_\tau(\epsilon(r,\tau)e^{-i\tau\langle r\rangle}) + \tau^{\kappa+1}h_\nu.
\end{aligned}
\tag{4.93}
$$

Applying R_τ to both sides of (4.93) then yields

$$
\begin{aligned}
E_{\kappa+1}^\nu &= R_\tau(\chi_{>|\tau|^{-1}}g_{\kappa+1}^\nu) + \sum_{a=1}^{\kappa}(\tau^a F_a e^{-i\tau\langle r\rangle}) + \tau R_\tau(\chi_{<|\tau|^{-1}}\zeta_\kappa^\nu) + \tau R_\tau(\zeta_\kappa^{\nu-3}e^{-i\tau\langle r\rangle})\\
&\quad + \tau^2 R_\tau(\zeta_{\kappa-1}^{\nu-2}e^{-i\tau\langle r\rangle}) + \sum_{m=3}^{\kappa}\tau^m R_\tau(\zeta_{\kappa+1-m}^\nu e^{-i\tau\langle r\rangle}) + \tau^\kappa\epsilon(r,\tau)e^{-i\tau\langle r\rangle} + \tau^{\kappa+1}(R_\tau h_\nu).
\end{aligned}
\tag{4.94}
$$

Part 1 of the proposition, (4.66), and (4.68) then give

$$
E_{\kappa+1}^\nu = R_\tau(\chi_{>|\tau|^{-1}}g_{\kappa+1}^\nu) + \sum_{m=1}^{\kappa}\Big(\tau^m(F_m + R_0\zeta_{\lambda-m}^{\nu-3m})e^{-i\tau\langle r\rangle}\Big) + \tau^\kappa\epsilon(r,\tau)e^{-i\tau\langle r\rangle} + \tau^{\kappa+1}(R_\tau h_{\nu-3\kappa-3})
\tag{4.95}
$$

as desired. This concludes the proof of the proposition. □

In the following lemma we prove the claim that (4.76) implies (4.77) used in Proposition 4.5.

Lemma 4.6. *Let φ^j be a family of functions indexed by j which satisfy*

$$(\partial_r + \frac{1}{r})\varphi^j = -\frac{1}{2}\Delta \sum_{k=a}^{j-1} \nabla^{j-1-k}\frac{x}{r}\varphi^k, \qquad j \geq a+1 \tag{4.96}$$

and $(\partial_r + \frac{1}{r})\varphi^a = 0$ for some fixed a. Then

$$\begin{aligned}(\partial_r + \frac{1}{r})&\sum_{k=a}^{j-1} \nabla^{j-1-k}\frac{x}{r}\varphi^k \\ &= -\frac{1}{2}\Delta \sum_{k=a+1}^{j-1} \nabla^{j-1-k}\frac{x}{r}\Big(\sum_{\ell=a}^{k-1}\nabla^{k-1-\ell}\frac{x}{r}\varphi^\ell\Big).\end{aligned} \tag{4.97}$$

Note that if we define

$$\varphi_1^j = \sum_{k=a}^{j-1} \nabla^{j-1-k}\frac{x}{r}\varphi^k$$

then the lemma shows that φ_1^j is a family of functions indexed by j which satisfy the assumptions of the lemma with a replaced by $a+1$. This fact is what allows us to use (4.78) in Proposition 4.5.

Proof. We have

$$\begin{aligned}[(\partial_r + \frac{1}{r}), \nabla^b] &= \sum_{\ell=1}^{b} \nabla^{b-\ell}[(\partial_r + \frac{1}{r}), \nabla]\nabla^{\ell-1} \\ &= -\frac{1}{2}\sum_{\ell=1}^{b} \nabla^{b-\ell}[\Delta, \frac{x}{r}]\nabla^{\ell-1}.\end{aligned} \tag{4.98}$$

Note we have

$$(\partial_r + \frac{1}{r})\varphi^a = 0 \tag{4.99}$$

by assumption and direct calculation shows $(\partial_r + \frac{1}{r})$ commutes with $\frac{x}{r}$. We find

$$
\begin{aligned}
&(\partial_r + \frac{1}{r})\sum_{k=a}^{j-1} \nabla^{j-1-k}\frac{x}{r}\varphi^k \\
&\qquad = \sum_{k=a+1}^{j-1} \nabla^{j-1-k}\frac{x}{r}(\partial_r + \frac{1}{r})\varphi^k + \sum_{k=a}^{j-2}[(\partial_r + \frac{1}{r}), \nabla^{j-1-k}]\frac{x}{r}\varphi^k \\
&\qquad = -\frac{1}{2}\sum_{k=a+1}^{j-1}\sum_{\ell=a}^{k-1} \nabla^{j-1-k}\Big(\frac{x}{r}\Delta\Big)\nabla^{k-1-\ell}\frac{x}{r}\varphi^\ell \\
&\qquad\quad - \frac{1}{2}\sum_{k=a}^{j-2}\sum_{\ell=1}^{j-1-k} \nabla^{j-1-k-\ell}[\Delta, \frac{x}{r}]\nabla^{\ell-1}\frac{x}{r}\varphi^k.
\end{aligned}
$$

The first equality uses (4.99). The second equality uses our assumption (4.96) and (4.98).

Changing the order of summation in the first term on the right hand side and simply switching the indexing labels in the second term yields

$$
\begin{aligned}
&(\partial_r + \frac{1}{r})\sum_{k=a}^{j-1} \nabla^{j-1-k}\frac{x}{r}\varphi^k \\
&\qquad = -\frac{1}{2}\sum_{\ell=a}^{j-2}\sum_{k=\ell+1}^{j-1} \nabla^{j-1-k}\Big(\frac{x}{r}\Delta\Big)\nabla^{k-1-\ell}\frac{x}{r}\varphi^\ell \\
&\qquad\quad - \frac{1}{2}\sum_{\ell=a}^{j-2}\sum_{k=1}^{j-1-\ell} \nabla^{j-1-\ell-k}[\Delta, \frac{x}{r}]\nabla^{k-1}\frac{x}{r}\varphi^\ell \\
&\qquad = -\frac{1}{2}\sum_{\ell=a}^{j-2}\sum_{k=\ell+1}^{j-1} \nabla^{j-1-k}\Big(\frac{x}{r}\Delta + [\Delta, \frac{x}{r}]\Big)\nabla^{k-1-\ell}\frac{x}{r}\varphi^\ell \\
&\qquad = -\frac{1}{2}\Delta\sum_{k=a+1}^{j-1}\sum_{\ell=a}^{k-1} \nabla^{j-1-k}\frac{x}{r}\nabla^{k-1-\ell}\frac{x}{r}\varphi^\ell \\
&\qquad = -\frac{1}{2}\Delta\sum_{k=a+1}^{j-1} \nabla^{j-1-k}\frac{x}{r}\sum_{\ell=a}^{k-1}\nabla^{k-1-\ell}\frac{x}{r}\varphi^\ell.
\end{aligned}
$$

To obtain the second equality we redefine the k index by $k \to k+\ell$. To obtain the third equality we switch the order of summation. This completes the proof of the lemma. $\square$

CHAPTER 5

Pointwise resolvent bounds

In this section we establish the pointwise resolvent bounds used in the proof of the main theorem. Our argument uses the Sobolev embedding

$$\|\phi\|_{L^\infty(\mathbb{S}^2)} \lesssim \|\phi\|_{L^2(\mathbb{S}^2)} + \|\Omega^2\phi\|_{L^2(\mathbb{S}^2)} \tag{5.1}$$

to obtain useful $L^2_r L^\infty_\omega(A_m)$ bounds on g and $R_\tau g$. For reference we begin with two preliminary lemmas resulting from a straightforward application of (5.1).

Lemma 5.1. *If ϕ satisfies*

$$\|\langle r\rangle^p\phi\|_{L^2(A_m)} + \sum_{|\alpha|=2}\|\langle r\rangle^p\Omega^\alpha\phi\|_{L^2(A_m)} \lesssim 1$$

then

$$2^{m(1+p)}\|\phi\|_{L^2_r L^\infty_\omega(A_m)} \lesssim 1.$$

Proof. A change of coordinates yields

$$\begin{aligned}\|\langle r\rangle^p\phi\|^2_{L^2(A_m)} &= \int_{r\approx 2^m}\int_\omega |\langle r\rangle^p\phi|^2 r^2\ dr d\omega\\ &\approx 2^{2m(1+p)}\|\phi\|^2_{L^2_r L^2_\omega(A_m)}.\end{aligned}$$

Then from the Sobolev embedding (5.1) we obtain

$$\begin{aligned}2^{m(1+p)}\|\phi\|_{L^2_r L^\infty_\omega(A_m)} &\lesssim 2^{m(1+p)}\left(\|\phi\|_{L^2_r L^2_\omega(A_m)} + \|\Omega^2\phi\|_{L^2_r L^2_\omega(A_m)}\right)\\ &\lesssim \|\langle r\rangle^p\phi\|_{L^2(A_m)} + \|\langle r\rangle^p\Omega^2\phi\|_{L^2(A_m)} \qquad (5.2)\\ &\lesssim 1.\end{aligned}$$

$\square$

Lemma 5.2. *If ϕ, $S_r\phi$, $\Omega^2\phi$, $\Omega^2 S_r\phi \in \mathcal{LE}^*$, then*

$$|\phi| \lesssim \langle r\rangle^{-2}.$$

Furthermore, if $\phi \in Z^{n,q}$, then

$$|\partial_r^p \phi| \lesssim \langle r\rangle^{-2-p-q}, \qquad p \le n-3.$$

Proof. By assumption, we have

$$\langle r\rangle^{\frac{1}{2}}\phi \in L^2(A_m) \qquad \textit{and} \qquad \langle r\rangle^{\frac{1}{2}}\Omega^2\phi \in L^2(A_m)$$

$$\langle r\rangle^{\frac{3}{2}}\partial_r\phi \in L^2(A_m) \qquad \textit{and} \qquad \langle r\rangle^{\frac{3}{2}}\Omega^2\partial_r\phi \in L^2(A_m).$$

Then by (5.1),

$$2^{\frac{3m}{2}}\|\phi\|_{L^2_r L^\infty_\omega(A_m)} \lesssim 1 \qquad \textit{and} \qquad 2^{\frac{5m}{2}}\|\partial_r\phi\|_{L^2_r L^\infty_\omega(A_m)} \lesssim 1.$$

Using the Fundamental Theorem of Calculus and Cauchy-Schwarz, we find pointwise bounds on ϕ:

$$\begin{aligned}\|\phi\|_{L^\infty(A_m)} &\lesssim 2^{-\frac{m}{2}}\|\phi\|_{L^2_r L^\infty_\omega(A_m)} + 2^{\frac{m}{2}}\|\partial_r\phi\|_{L^2_r L^\infty_\omega(A_m)} \\ &\lesssim 2^{-2m}.\end{aligned}$$

Thus $|\phi| \lesssim \langle r\rangle^{-2}$.

Now assume $\phi \in Z^{n,q}$. We can write $\langle r\rangle^p \partial_r^p S_r^p$ as a linear combination of S_r^k for $k \le p$ so

$$|\langle r\rangle^{q+p}\partial_r^p\phi| \lesssim \sum_{k\le p} |\langle r\rangle^q S_r^k\phi|.$$

Since $[\Omega, r] = 0$ and $[S_r, r] = r$, it follows by the definitions of $Z^{n,q}$ that

$$\Omega^j S_r^k \langle r\rangle^{q+p}\partial_r^p\phi \in \mathcal{LE}^*, \quad j+k \le 3$$

if $p \leq n-3$. Therefore by the first part of the proposition we have

$$|\langle r\rangle^{q+p}\partial_r^p\phi| \lesssim \langle r\rangle^{-2}.$$

$\square$

In Proposition 5.3 we will use notation as in Proposition 2.5 so that M indicates the regularity assumed for g. We again take $v_{ijk} = T^i\Omega^j S^k v$ and $g_{ijk} = T^i\Omega^j S^k g$. Similarly, we write $v_{<i<j<k} = T^{<i}\Omega^{<i}S^{<k}v$ and use analogous notation for g.

Proposition 5.3. *Assume* $\Im\tau \leq 0$. *Let* $g \in \mathcal{LE}^*$ *satisfy* (2.26) *and possibly depend on* τ. *Set* $v = R_\tau g$.

1. *If* $|\tau| \gtrsim 1$, *then*
$$|T^i\Omega^j S^k v(\tau)| \lesssim (|\tau|\langle r\rangle)^{-1}, \quad i+4j+16k \leq M-20. \tag{5.3}$$

2. *If* $|\tau| \lesssim 1$, *then*
$$|T^i\Omega^j S^k v(\tau)| \lesssim \begin{cases} \min\{1, (|\tau|\langle r\rangle)^{-1}\} & i=0 \\ \langle r\rangle^{-1} & i\geq 1 \end{cases} \qquad i+4j+16k \leq M-20. \tag{5.4}$$

3. *If* $\tau \in \mathbb{R}\setminus\{0\}$, *then we have the outgoing radiation condition:*
$$\lim_{|x|\to\infty} r(\partial_r + i\tau)T^i\Omega^j S^k v(\tau) = 0, \quad i+4j+16k \leq M-20. \tag{5.5}$$

Proof. As before, we write $g_{ijk} := T^i\Omega^j S^k g$ and $v_{ijk} := T^i\Omega^j S^k v$. Note since we allow for the possibility that g depends on τ our assumption that $S^k g = (r\partial_r - \tau\partial_\tau)^k g$ is in $\mathcal{LE}^*$ does not translate to $\mathcal{LE}^*$ bounds on $S_r g$.

To begin we collect results that are independent of the size of τ. We claim the following hold for all τ such that $\Im\tau \leq 0$.

1. The estimate
$$\sum_m 2^{\frac{m}{2}}\|(\partial_r^2+\tau^2)(rv_{ijk})\|_{L^2_r L^\infty_\omega(A_m)} \lesssim 1 \tag{5.6}$$

implies

$$|v_{ijk}| \lesssim (\langle r\rangle|\tau|)^{-1} \qquad \text{and} \qquad |\partial_r v_{ijk}| \lesssim \langle r\rangle^{-1}. \tag{5.7}$$

2. We have the following expression for $(\partial_r^2+\tau^2)(rv_{ijk})$

$$(\partial_r^2+\tau^2)rv_{ijk} = -r^{-1}\Delta_\omega v_{ijk} + r(Q_\ell+Q_r)v_{\leq i\leq j\leq k} + rg_{\leq i\leq j\leq k} \tag{5.8}$$

where Q_ℓ, Q_r are as in Proposition 2.5.

3. The functions g_{ijk} and v_{ijk} satisfy

$$\sum_m \|\langle r\rangle^{\frac{1}{2}} g_{ijk}\|_{L^2(A_m)} \lesssim 1, \qquad i+4j+16k \leq M \tag{5.9}$$

and

$$\|(\langle r\rangle^{-1}+|\tau|)v_{ijk}\|_{\mathcal{LE}} + \|\nabla v_{ijk}\|_{\mathcal{LE}} + \|(\langle r\rangle^{-1}+|\tau|)^{-1}\nabla^2 v_{ijk}\|_{\mathcal{LE}} \lesssim 1, \qquad i+4j+16k \leq M-4. \tag{5.10}$$

For claim 3, we see (5.9) is simply a restatement of the assumption $g_{ijk} \in \mathcal{LE}^*$. When $\Im\tau < 0$, (5.10) follows by Proposition 2.5. When $\tau \in \mathbb{R}$, we have the same result by Corollary 2.7.

To establish claim 2 we write

$$(\partial_r^2+\tau^2) = P_\tau - (2r^{-1}\partial_r + r^{-2}\Delta_\omega + Q_\ell + Q_r)$$

and calculate

$$\begin{aligned}(\partial_r^2+\tau^2)rv_{ijk} &= r(\partial_r^2+\tau^2)v_{ijk} + 2\partial_r v_{ijk}\\ &= rP_\tau v_{ijk} - r^{-1}\Delta_\omega v_{ijk} - r(Q_\ell+Q_r)v_{ijk}.\end{aligned}$$

Then (5.8) follows from rewriting the first term using (2.30).

Finally to prove claim 1, assume (5.6) holds. The pointwise bounds on v_{ijk} and $\partial_r v_{ijk}$ shall follow using the fundamental solution for $(\partial_r^2+\tau^2)$, which is given by $K_\tau(s) = \tau^{-1}e^{-i\tau|s|}$. Note that when $|\tau| \lesssim 1$ the desired result (5.7) provides better bounds for $\partial_r v_{ijk}$ than v_{ijk}. We will use this improvement to establish (5.4) for $i \geq 1$.

We have the following pointwise bounds for $K_\tau(s)$:

$$|K_\tau(s)| \lesssim |\tau|^{-1}, \quad |\partial_s K_\tau(s)| \lesssim 1.$$

Writing $rv_{ijk} = \psi * K_\tau(s)$, where

$$\psi(r) = \sup_{\omega \in \mathbb{S}^2} (\partial_r^2 + \tau^2)(rv_{ijk})$$

satisfies $\sum_m 2^{\frac{m}{2}} \|\psi\|_{L_r^2(A_m)} \lesssim 1$ by assumption, and applying Schwarz' inequality yields

$$\begin{aligned}
|\tau||rv_{ij\ell}| &\lesssim |\tau| \sum_m \left| \int_{A_m} \psi(s) K_\tau(r-s)\ ds \right| \\
&\leq |\tau||\tau|^{-1} \sum_m 2^{\frac{m}{2}} \|\psi\|_{L_r^2(A_m)} \\
&\lesssim 1.
\end{aligned}$$

Thus we have

$$|\tau||v_{ijk}| \lesssim \langle r \rangle^{-1} \tag{5.11}$$

as desired.

Similarly, now using $\partial_s K_\tau(s)$, we find

$$\begin{aligned}
|\partial_r(rv_{ijk})| &= \sum_m \left| \int_{A_m} \psi(s) \partial_r K_\tau(r-s)\ ds \right| \\
&\leq \sum_m 2^{\frac{m}{2}} \|\psi\|_{L_r^2(A_m)} \\
&\lesssim 1
\end{aligned}$$

so that

$$|\partial_r v_{ij\ell}| \lesssim \langle r \rangle^{-1}. \tag{5.12}$$

as desired. This concludes the proof that (5.6) implies (5.7).

We provide a brief summary of the arguments below. We will use claim 1 to establish the desired pointwise bounds for large τ and for the case that $\langle r\rangle \gtrsim |\tau|^{-1}$ when τ is small. To show (5.6) holds, we apply Lemma 5.1 to (5.9) and (5.10) and obtain favorable bounds on each term on the right hand side of (5.8). Note that when $|\tau| \gtrsim 1$, we have the advantage that

$$\|\langle r\rangle^{-\frac{1}{2}} v_{ijk}\|_{L^2(A_m)} \lesssim \|\langle r\rangle^{-\frac{1}{2}}(\langle r\rangle^{-1} + |\tau|) v_{ijk}\|_{L^2(A_m)}.$$

The right hand side is bounded by (5.10) regardless of the size of τ. When $|\tau| \lesssim 1$, we are stuck with either a $|\tau|$ factor or an $\langle r\rangle^{-1}$ factor on the left hand side of the above estimate. In the region $\langle r\rangle \gtrsim |\tau|^{-1}$, the summation in (5.6) is limited to $m > \log|\tau|^{-1}$, which allows us to handle the $|\tau|$ factor. In the region $\langle r\rangle \lesssim |\tau|^{-1}$, we see that claim 1 is insufficient for bounding $|v_{ijk}|$ since $(|\tau|\langle r\rangle)^{-1}$ is unbounded, so a different strategy will be needed. The advantage we have here is that the $(\langle r\rangle^{-1} + |\tau|)^{-1}$ weight in the second order term of (5.10) is bounded below by $\langle r\rangle$ when $\langle r\rangle \lesssim |\tau|^{-1}$.

1. Large $|\tau|$: To prove the bounds for large τ we establish (5.6) by bounding each term on the right hand side of (5.8) separately. We denote

$$\phi_1 = r^{-1}\Delta_\omega v_{ijk}, \qquad \phi_2 = r(Q_\ell + Q_r) v_{\le i \le j \le k}, \qquad \phi_3 = r g_{\le i \le j \le k} \tag{5.13}$$

so we have

$$\sum_m 2^{\frac{m}{2}} \|(\partial_r^2 + \tau^2)(r v_{ijk})\|_{L^2_r L^\infty_\omega(A_m)} \lesssim \sum_m 2^{\frac{m}{2}} \Big(\|\phi_1\|_{L^2_r L^\infty_\omega(A_m)} + \|\phi_2\|_{L^2_r L^\infty_\omega(A_m)} + \|\phi_3\|_{L^2_r L^\infty_\omega(A_m)} \Big).$$

Using (5.2) in Lemma 5.1, it suffices to show

$$\sum_m \|\langle r\rangle^{-\frac{1}{2}} \phi_n\|_{L^2(A_m)} + \|\langle r\rangle^{-\frac{1}{2}} \Omega^2 \phi_n\|_{L^2(A_m)} \lesssim 1 \tag{5.14}$$

for $1 \le n \le 3$ to establish (5.6). The pointwise bounds for large τ then follow by claim 1. For ϕ_3, it follows immediately from (5.9) that (5.14) holds when $i + 4j + 16k \le M - 8$. For ϕ_1 and ϕ_2 we shall use (5.10).

When $|\tau| \gtrsim 1$, (5.10) implies

$$\|\langle r\rangle^{-\frac{1}{2}} v_{ijk}\|_{L^2(A_m)} \lesssim |\tau| \|\langle r\rangle^{-\frac{1}{2}} v_{ijk}\|_{L^2(A_m)} \lesssim 1 \qquad i+4j+16k < M-4. \tag{5.15}$$

For ϕ_1 we replace Δ_ω by $\sum_{|\alpha|=2} \Omega^\alpha$ and use (5.15) to find

$$\begin{aligned}\sum_m \|\langle r\rangle^{-\frac{1}{2}} \phi_1\|_{L^2(A_m)} + \|\langle r\rangle^{-\frac{1}{2}} \Omega^2 \phi_1\|_{L^2(A_m)} &= \sum_m \|\langle r\rangle^{-\frac{3}{2}} v_{i(j+2)k}\|_{L^2(A_m)} + \|\langle r\rangle^{-\frac{3}{2}} v_{i(j+4)k}\|_{L^2(A_m)} \\ &\lesssim \sum_m 2^{-\frac{3m}{2}} \|v_{i(j+4)k}\|_{L^2(A_m)} \\ &\lesssim 1, \qquad i+4j+16k \le M-20.\end{aligned}$$

For ϕ_2 we first establish the allowable weights to obtain the desired summability. Since we are concerned only with what function space ϕ_2 is in, we consider only the symbol classes of the coefficients of Q_ℓ and Q_r. As before, we use ρ_ℓ^q to denote a representative of the symbol class $\ell^1 S(r^q)$ and ρ_r^q to denote a representative of the symbol class $S_{rad}(r^q)$. We find

$$\begin{aligned}&\textstyle\sum_m \|\langle r\rangle^{-\frac{1}{2}} \rho_\ell^q v_{ijk}\|_{L^2(A_m)} \lesssim 1, \qquad \text{if } q \le 0 \\ &\textstyle\sum_m \|\langle r\rangle^{-\frac{1}{2}} \rho_r^q v_{ijk}\|_{L^2(A_m)} \lesssim 1, \qquad \text{if } q < 0\end{aligned}$$

for $i+4j+16k \le M-4$. This tells us that if the coefficients of $r(Q_\ell + Q_r)$ are in $\ell^1 S(r^q)$ with $q \le 0$ or in $S_{rad}(r^q)$ with $q < 0$, then (5.14) holds for ϕ_2. Recall from (2.28) and (2.29) that

$$Q_\ell + Q_r = |\tau|(\rho_\ell^{-\kappa}\partial_i + \rho_\ell^{-\kappa-1}) + \rho_\ell^{-\kappa}\partial_i\partial_j + \rho_\ell^{-\kappa-1}\partial_i + \rho_\ell^{-\kappa-2} + \rho_r^{-\kappa-2}\Delta_\omega + \rho_r^{-\kappa-2}$$

so the coefficients of $r(Q_\ell + Q_r)$ are seen to be favorable when $\kappa \ge 0$ once we note that the $|\tau|$ factor in the first term is harmless due to the $|\tau|$ factor in (5.15). Viewing the derivatives in $Q_\ell + Q_r$ as vector fields and replacing Δ_ω as we did for ϕ_1, we find (5.14) holds for ϕ_2 when $i+4j+16k \le M-20$.

Combining the results for each ϕ_n and applying claim 1, we obtain

$$|v_{ijk}| \lesssim (|\tau|\langle r\rangle)^{-1}, \qquad i+4j+16k \le M-20.$$

2. Small $|\tau|$: First we consider the case $2^m \gtrsim |\tau|^{-1}$. In this region we proceed in the same manner as for the proof of the large τ bounds. In other words we will show (5.6) holds by establishing

(5.14) for ϕ_n as in (5.13). The estimate (5.9) still directly implies that (5.14) holds for ϕ_3 when $i+4j+16k \leq N-8$. Since τ is small we keep the τ factor in the lowest order term in (5.10) so that we have

$$|\tau|\|\langle r\rangle^{-\frac{1}{2}} v_{ijk}\|_{L^2(A_m)} \lesssim 1 \tag{5.16}$$

and

$$\|\langle r\rangle^{-\frac{1}{2}} \nabla v_{ijk}\|_{L^2(A_m)} \lesssim 1 \tag{5.17}$$

when $i+4j+16k \leq M-4$. First consider ϕ_2, which we write as

$$|\tau|\big(\rho_\ell^{-\kappa+1} v_{(i+1)jk} + \rho_\ell^{-\kappa} v_{ijk}\big) + \rho_\ell^{-\kappa+1} \nabla v_{(i+1)jk} + \rho_\ell^{-\kappa} \nabla v_{ijk} + \rho_\ell^{-\kappa-1} v_{ijk} + \rho_r^{-\kappa-1} v_{i(j+2)k} + \rho_r^{-\kappa-1} v_{ijk}.$$

For the sake of exposition we have included only the v_{ijk} piece of ϕ_2 and note that our results apply for $v_{<i<j<k}$. For the first two terms we use (5.16) to obtain (5.14) when $i+4j+16k \leq M-13$. For the second two terms we use (5.17) to obtain (5.14) when $i+4j+16k \leq M-13$. The remaining terms each come with a weight better than r^{-1}. We handle these along with $\phi_1 = r^{-1} v_{i(j+2)k}$ by showing that the r^{-1} weight is sufficient to obtain the desired summability. By (5.10) we have $|\tau|\|\langle r\rangle^{-\frac{1}{2}} v_{ijk}\|_{L^2(A_m)} \lesssim 1$, which allows us to calculate

$$\sum_{m \geq \log|\tau|^{-1}} \|\langle r\rangle^{-\frac{3}{2}} v_{ijk}\|_{L^2(A_m)} \lesssim \sum_{m \geq \log|\tau|^{-1}} 2^{-m}|\tau|^{-1} \lesssim 1$$

so (5.14) holds for ϕ_1 and the remaining terms in ϕ_2 for $i+4j+16k \leq M-20$. It follows by claim 1 and Lemma 5.1 that when $|\tau| \lesssim 1$ we have

$$|v_{ijk}| \lesssim (|\tau|\langle r\rangle)^{-1} \qquad \text{and} \qquad |\partial_r v_{ijk}| \lesssim \langle r\rangle^{-1}, \qquad i+4j+16k \leq M-20$$

in the region $\langle r\rangle \gtrsim |\tau|^{-1}$.

We now consider the case $\langle r\rangle \lesssim |\tau|^{-1}$. In this region we have

$$\langle r\rangle \lesssim (\langle r\rangle^{-1} + |\tau|)^{-1}$$

so Proposition 5.1 applied to (5.10) produces the inequality

$$2^{-m}\|v_{ijk}\|_{L^2_rL^\infty_\omega(A_m)} + \|\nabla v_{ijk}\|_{L^2_rL^\infty_\omega(A_m)} + 2^m\|\nabla^2 v_{ijk}\|_{L^2_rL^\infty_\omega(A_m)} \lesssim 2^{-\frac{m}{2}}$$

for $i+4j+16k \le M-12$.

We write $v^\omega_{ijk} = \sup_{\omega\in\mathbb{S}^2}|v_{ijk}|$ and calculate

$$\begin{aligned}
\|v^\omega_{ijk}\|_{L^\infty(A_m)} &= \left\|\int_0^s \partial_r\left(\chi_{\approx m}(r)v^\omega_{ijk}(r)\right)\, dr\right\|_{L^\infty(A_m)} \\
&\lesssim \int_0^\infty 2^{-m}\left|\chi'_{\approx 1}\left(\frac{r}{2^m}\right)v^\omega_{ijk}\right|\, dr + \int_0^\infty \left|\chi_{\approx m}(r)\partial_r v^\omega_{ijk}\right|\, dr \\
&\lesssim 2^{-m}2^{\frac{m}{2}}\|v^\omega_{ijk}\|_{L^2_r(\tilde{A}_m)} + 2^{\frac{m}{2}}\|\partial_r v^\omega_{ijk}\|_{L^2_r(\tilde{A}_m)} \\
&\lesssim 1.
\end{aligned}$$

Here we use $\tilde{A}_m$ to indicate integration over a region slightly altered from A_m due to the cutoff function. Similarly, we have

$$\|\partial_r v^\omega_{ijk}(s)\|_{L^\infty(A_m)} \lesssim 2^{-\frac{m}{2}}\|\partial_r v^\omega_{ijk}\|_{L^2_r(A_m)} + 2^{\frac{m}{2}}\|\partial_r^2 v^\omega_{ijk}\|_{L^2_r(A_m)} \lesssim 2^{-m}$$

so $|\partial_r v_{0jk}| \lesssim \langle r\rangle^{-1}$. We summarize the above results for $|\tau| \lesssim 1$ in the region $\langle r\rangle \lesssim |\tau|^{-1}$:

$$|v_{ijk}| \lesssim 1 \quad \text{and} \quad |\partial_r v_{ijk}| \lesssim \langle r\rangle^{-1}, \qquad i+4j+16k \le M-12. \tag{5.18}$$

Finally, if $i \ge 1$ we write

$$|v_{ijk}| = |Tv_{(i-1)jk}| \lesssim |r^{-1}S_r v_{(i-1)jk}| + |r^{-1}\Omega v_{(i-1)jk}| \lesssim \langle r\rangle^{-1}.$$

This concludes the proof of (5.4).

3. Real τ: Consider (5.8) and note that we have

$$(\partial_r - i\tau)(\partial_r + i\tau)(rv_{ijk}) = (\partial_r^2 + \tau^2)(rv_{ijk})$$

so that

$$\partial_r\big(e^{-i\tau r}(\partial_r + i\tau)(rv_{ijk})\big) = (\partial_r^2 + \tau^2)(rv_{ijk})e^{-i\tau r}. \tag{5.19}$$

If $(\partial_r^2 + \tau^2)(rv_{ijk})$ is integrable in r, then the limit $\lim_{|x|\to\infty} |(\partial_r + i\tau)(rv_{ijk})|$ exists since

$$\lim_{|x|\to\infty} |(\partial_r + i\tau)(rv_{ijk})| = \Big|\int_0^\infty (\partial_r^2 + \tau^2)(rv_{ijk})e^{-i\tau r}\, dr + v_{ijk}(0)\Big|.$$

To see $(\partial_r^2 + \tau^2)(rv_{ijk})$ is integrable in r, note that (5.6) implies

$$\sum_m \int_{A_m} \|(\partial_r^2 + \tau^2)(rv_{ijk})\|_{L^\infty_\omega}\, dr \lesssim \sum_m 2^{\frac{m}{2}} \|(\partial_r^2 + \tau^2)(rv_{ijk})\|_{L^2_r L^\infty_\omega(A_m)} \lesssim 1.$$

We established (5.6) for large τ and for $\langle r\rangle \gtrsim |\tau|^{-1}$ when $|\tau| \lesssim 1$. This leaves the case of $|\tau| \lesssim 1$ and $\langle r\rangle \lesssim |\tau|^{-1}$. We write

$$(\partial_r^2 + \tau^2)(rv_{ijk}) = r\partial_r^2 v_{ijk} + 2\partial_r v_{ijk} + \tau^2 r v_{ijk}$$

and use the fact that $|\tau r| \lesssim 1$ to calculate

$$\begin{aligned}
&\int \|(\partial_r^2 + \tau^2)(rv_{ijk})\|_{L^\infty_\omega}\, dr \\
&\quad\lesssim \int \|r\partial_r^2 v_{ijk}\|_{L^\infty_\omega} + \|\partial_r v_{ijk}\|_{L^\infty_\omega} + |\tau|\|v_{ijk}\|_{L^\infty_\omega}\, dr \\
&\quad\lesssim \sum_m 2^{\frac{m}{2}} \left(2^m \|\partial_r^2 v_{0jk}\|_{L^2_r L^\infty_\omega(A_m)} + \|\partial_r v_{0jk}\|_{L^2_r L^\infty_\omega(A_m)} + |\tau|\|v_{0jk}\|_{L^2_r L^\infty_\omega(A_m)}\right).
\end{aligned}$$

The terms on the right hand side are bounded by applying Lemma 5.1 and (5.10). Thus $(\partial_r^2+\tau^2)(rv_{ijk})$ is integrable and the limit $\lim_{|x|\to\infty}(\partial_r + i\tau)(rv_{ijk})$ exists.

Take

$$c = \lim_{|x|\to\infty} |(\partial_r + i\tau)(rv_{ijk})| = \lim_{|x|\to\infty} |r(\partial_r + i\tau)v_{ijk} + v_{ijk}|.$$

By parts 1 and 2 of this proposition, $\lim_{r\to\infty} v_{ijk} = 0$ so that $\lim_{r\to\infty} |r(\partial_r + i\tau)v_{ijk}| = c$. Thus we can write $|(\partial_r + i\tau)v_{ijk}| = cr^{-1} + o(r^{-1})$.

On the other hand, since τ is real, we have by Corollary 2.7 that v_{ijk} satisfies the radiation condition:

$$\lim_{m\to\infty} 2^{-\frac{m}{2}} \|(\partial_r + i\tau)v_{ijk}\|_{L^2(A_m)} = 0. \tag{5.20}$$

Applying Lemma 5.1 to (5.20) yields

$$\lim_{m\to\infty} 2^{\frac{m}{2}} \|(\partial_r + i\tau)v_{ijk}\|_{L^2_r L^\infty_\omega} = 0. \tag{5.21}$$

It follows that

$$0 = \lim_{m\to\infty} 2^{\frac{m}{2}} \|cr^{-1} + o(r^{-1})\|_{L^2_r(A_m)} = \lim_{m\to\infty} 2^{\frac{m}{2}} c2^{-\frac{m}{2}} = c$$

Thus $(\partial_r + i\tau)v_{ijk} \in o(r^{-1})$, which concludes the proof of (5.5).

□

Ultimately our goal is to use the fact that if u solves the homogeneous Cauchy problem with initial data $u(0,\cdot) = u_0$ and $\partial_t u(0,\cdot) = u_1$ then

$$\hat{u}(\tau) = R_\tau(-i\tau u_0 + P^1 u_0 + u_1)$$

to invert the Fourier transform and obtain decay in t. Inverting the Fourier transform introduces time decay when we write $e^{it\tau} = \partial_\tau\left(\frac{e^{it\tau}}{it}\right)$ and integrate by parts. Therefore we need pointwise bounds on $\partial_\tau R_\tau g$.

We will use the results of Proposition 5.3 to establish pointwise bounds on $(\tau\partial_\tau)^p(ve^{ir\tau})$. Note

$$\tau\partial_\tau(ve^{ir\tau}) = [(-S + r(\partial_r + i\tau))v]e^{ir\tau}.$$

This motivates the following lemma, which will be used to prove the subsequent proposition stating the pointwise bounds on $(\tau\partial_\tau)^p(ve^{ir\tau})$ for $|\tau| \gtrsim 1$. We remark that while the above calculation shows we are primarily concerned with $(\partial_r + i\tau)^p v_{00k}$, our methods will generate T and Ω vector fields as we induct in k, so we handle $(\partial_r + i\tau)^p v_{ijk}$.

Lemma 5.4. *Let $g \in Z^{n,q}$. If $\tau \in \mathbb{R}$ and $|\tau| \gtrsim 1$, then $v = R_\tau g$ satisfies the pointwise bounds*

$$|(\partial_r + i\tau)^p v_{ijk}| \lesssim |\tau|^{p-1}\langle r\rangle^{-p-1} \qquad p \le q, \quad p \le n-3, \qquad \textit{and } i + 4j + 16k \le n - 20 - 8p. \tag{5.22}$$

Proof. Note $i + j + k < i + 4j + 16k < n$ so $g \in Z^{n,q}$ implies g satisfies (2.26) with $M = n$, so the results of Proposition 5.3 apply with $M = n$.

When $p = 0$, (5.22) follows from (5.3).

To handle $p = 1$, we find

$$(\partial_r^2 + \tau^2)(rv_{ijk}) = -r^{-1}\Delta_\omega v_{ijk} + rQ_\ell v_{\leq i \leq j \leq k} + rQ_r v_{\leq i \leq j \leq k} + rg_{\leq i \leq j \leq k}.$$

The first three terms on the right hand side are pointwise bounded by $\langle r\rangle^{-2}$ using (5.3) and the fact that $\kappa \geq 2$. As usual we replace $\Delta_\omega v_{ijk}$ by $v_{i(j+2)k}$ so that the bounds hold when $i+4j+16k \leq n-28$. For the final term, we note that by Lemma 5.2, we have

$$|\partial_r^k g| \lesssim \langle r\rangle^{-2-k-q}, \qquad k \leq n-3$$

so $|rg| \lesssim \langle r\rangle^{-2}$ since $1 = p \leq q$. Thus we have

$$|(\partial_r^2 + \tau^2)(rv_{ijk})| \lesssim \langle r\rangle^{-2}. \tag{5.23}$$

Rewriting the left hand side of (5.23) we see

$$|(\partial_r - i\tau)(\partial_r + i\tau)(rv_{ijk})| \lesssim \langle r\rangle^{-2}. \tag{5.24}$$

By the radiation condition (5.5) and (5.3), we have $\lim_{r\to\infty}(\partial_r + i\tau)(rv_{ijk}) = 0$, so we can write (5.24) as in (5.19) and integrate (5.19) from infinity to find

$$|(\partial_r + i\tau)(rv_{ijk})| \lesssim \int_r^\infty |\langle s\rangle^{-2}| ds = \langle r\rangle^{-1}.$$

This allows us to calculate

$$r|(\partial_r + i\tau)v_{ijk}| = |(\partial_r + i\tau)(rv_{ijk}) - v_{ijk}| \lesssim \langle r\rangle^{-1}$$

so that $|(\partial_r + i\tau)v_{ijk}| \lesssim \langle r\rangle^{-2}$, as desired.

We proceed by induction. Fix p and assume we have

$$|(\partial_r + i\tau)^a v_{ijk}| \lesssim |\tau|^{a-1}\langle r\rangle^{-a-1}$$

for $a < p$ when $i + 4j + 16 \le n - 20 - 8a$. Note the following relations, which are obtained by direct calculation

$$\begin{gathered}(\partial_r + i\tau)^p(rv) = r(\partial_r + i\tau)^p v + C(\partial_r + i\tau)^{p-1}v,\\ (\partial_r + i\tau)^p(r^{-1}\Delta_\omega v) = \textstyle\sum_{m=0}^{p}(-1)^{p-m}c_m r^{-(1+p-m)}(\partial_r + i\tau)^m \Delta_\omega v.\end{gathered}$$

Using the above relations, we find

$$\begin{aligned}&(\partial_r - i\tau)(\partial_r + i\tau)^p(rv_{ijk})\\ &\quad= (\partial_r + i\tau)^{p-1}(-r^{-1}\Delta_\omega v_{ijk} + rQ_\ell v_{\le i\le j\le k} + rQ_r v_{\le i\le j\le k} + rg_{\le i\le j\le k})\\ &\quad= \sum_{m=0}^{p-1}\Big((-1)^{p-m+1}c_m r^{-(p-m)}(\partial_r + i\tau)^m \Delta_\omega v_{ijk}\Big) + r(\partial_r + i\tau)^{p-1}Q_\ell v_{\le i\le j\le k}\\ &\qquad\quad + C(\partial_r + i\tau)^{p-2}Q_\ell v_{\le i\le j\le k} + r(\partial_r + i\tau)^{p-1}Q_r v_{\le i\le j\le k} + C(\partial_r + i\tau)^{p-2}Q_r v_{ijk}\\ &\qquad\quad + r(\partial_r + i\tau)^{p-1}g_{\le i\le j\le k} + C(\partial_r + i\tau)^{p-2}g_{\le i\le j\le k}.\end{aligned} \tag{5.25}$$

We claim that each term on the right hand side of (5.25) is bounded by $|\tau|^{p-1}\langle r\rangle^{-p-1}$. For the first term, we have by the inductive hypothesis

$$|r^{-(p-m)}(\partial_r + i\tau)^m v_{i(j+2)k}| \lesssim |\tau|^{m-1}\langle r\rangle^{-p-1} \lesssim |\tau|^{p-1}\langle r\rangle^{-p-1}$$

when $i + 4j + 16k \le n - 20 - 8p$.

For the Q_ℓ and Q_r terms, we commute $(\partial_r + i\tau)$ with the coefficients of the operators and view the derivatives as vector fields. Direct calculation yields

$$\begin{aligned}&|(\partial_r + i\tau)^{p-1}Q_\ell v_{ijk}|\\ &\qquad\lesssim |\langle r\rangle^{-\kappa}(\partial_r + i\tau)^{p-1}v_{(i+2)jk}| + (1+|\tau|)|\langle r\rangle^{-\kappa}(\partial_r + i\tau)^{p-1}v_{(i+1)jk}|\\ &\qquad\quad + (1+|\tau|)|\langle r\rangle^{-\kappa-1}(\partial_r + i\tau)^{p-1}v_{ijk}| + |[(\partial_r + i\tau)^{p-1}, \rho_\ell^{-\kappa}]|\Big(|v_{(i+2)jk}| + |\tau v_{(i+1)jk}|\Big)\\ &\qquad\quad + |[(\partial_r + i\tau)^{p-1}, \rho_\ell^{-\kappa-1}]|\Big(|v_{(i+1)jk}| + |\tau v_{ijk}|\Big) + |[(\partial_r + i\tau)^{p-1}, \rho_\ell^{-\kappa-2}]v_{(i+1)jk}|\end{aligned} \tag{5.26}$$

and

$$
\begin{aligned}
|(\partial_r + i\tau)^{p-1} Q_r v_{ijk}| \lesssim |\rho_r^{-\kappa-2}(\partial_r + i\tau)^{p-1} v_{i(j+2)k}| + |\rho_r^{-\kappa-2}(\partial_r + i\tau)^{p-1} v_{ijk}| \\
+ \Big|[(\partial_r + i\tau)^{p-1}, \rho_r^{-\kappa-2}]\Big|\Big(|v_{i(j+2)k}| + |v_{ijk}|\Big).
\end{aligned} \tag{5.27}
$$

The Q_ℓ and Q_r terms are then bounded by $|\tau|^{p-1}\langle r\rangle^{-p-1}$ using our assumption $\kappa \geq 2$ once we note

$$
\begin{aligned}
[(\partial_r + i\tau)^{p-1}, \rho_\ell^{-\kappa}] &= \sum_{m=1}^{p-1} c_m \rho_\ell^{-\kappa-m}(\partial_r + i\tau)^{p-1-m} \\
[(\partial_r + i\tau)^{p-1}, \rho_r^{-\kappa}] &= \sum_{m=1}^{p-1} c_m \rho_r^{-\kappa-m}(\partial_r + i\tau)^{p-1-m}.
\end{aligned} \tag{5.28}
$$

For the last two terms in (5.25) we use Lemma 5.2 (which requires our assumption $p \leq n-3$) to find

$$
\begin{aligned}
|r(\partial_r + i\tau)^{p-1} g_{\leq i\leq j\leq k}| &+ |(\partial_r + i\tau)^{p-2} g_{\leq i\leq j\leq k}| \\
&= \Big|r \sum_{m=0}^{p-1} c_m (i\tau)^m \partial_r^{p-1-m} g_{\leq i\leq j\leq k}\Big| + \Big|\sum_{m=0}^{p-2} (i\tau)^m c_m \partial_r^{p-2-m} g_{\leq i\leq j\leq k}\Big| \\
&\lesssim \sum_{m=0}^{p-1} |\tau|^m \langle r\rangle^{1-2-p+1+m-q} + \sum_{m=0}^{p-2} |\tau|^m \langle r\rangle^{-2-p+2+m-q} \\
&\lesssim |\tau|^{p-1}\langle r\rangle^{-1-q} \\
&\lesssim |\tau|^{p-1}\langle r\rangle^{-1-p}.
\end{aligned}
$$

The last inequality holds since we assume $p \leq q$.

Now we have

$$
|(\partial_r - i\tau)(\partial_r + i\tau)^p (r v_{ijk})| \lesssim |\tau|^{p-1}\langle r\rangle^{-p-1}
$$

when $p \leq q$. Integrating, we find

$$
|(\partial_r + i\tau)^p (r v_{ijk})| \lesssim |\tau|^{p-1}\langle r\rangle^{-p}
$$

so that

$$|(\partial_r + i\tau)^p(v_{ijk})| = |r^{-1}(\partial_r + i\tau)^p(rv_{ijk}) - Cr^{-1}(\partial_r + i\tau)^{p-1}v_{ijk}| \lesssim |\tau|^{p-1}\langle r\rangle^{-p-1}.$$

To justify integrating from infinity we note

$$(\partial_r + i\tau)v_{ijk} = \langle r\rangle^{-1}x \cdot \nabla v_{ijk} + i\tau v$$

and thus

$$|(\partial_r + i\tau)^2 v_{ijk}| = |x\langle r\rangle^{-1}(\partial_r + i\tau)v_{(i+1)jk} + i\tau(\partial_r + i\tau)v| \lesssim |(\partial_r + i\tau)v_{(i+1)jk}| + |\tau||(\partial_r + i\tau)v|.$$

Iterating, we find

$$|(\partial_r + i\tau)^p v_{ijk}| \lesssim \sum_{m=0}^{p-1} |\tau|^m |(\partial_r + i\tau)v_{(i+p-1-m)jk}|$$

and thus $r(\partial_r + i\tau)v_{ijk} \to 0$ as $r \to \infty$ for $i + 4j + 16k \leq n - 20 - (p-1)$, which is satisfied since $n - 20 - 8p < n - 19 - p$. The argument allowing us to integrate in the $p = 1$ case thus applies to the general case. This concludes the proof of (5.22). □

Proposition 5.5. *Let $g \in Z^{n,q}$. If $\tau \in \mathbb{R}$ and $|\tau| \gtrsim 1$, then $v = R_\tau g$ satisfies the pointwise bounds*

$$|(\tau\partial_\tau)^p \left(ve^{i\tau\langle r\rangle}\right)| \lesssim |\tau|^{p-1}\langle r\rangle^{-1}, \qquad p \leq q \quad \textit{and} \quad 16p \leq n - 20. \tag{5.29}$$

Proof. Recall we defined $\langle r\rangle$ to be equal to r for $\langle r\rangle \geq 2$ (and $\langle r\rangle \gtrsim 1$), so it suffices to prove the proposition for $(\tau\partial_\tau)^p \left(ve^{i\tau r}\right)$ since (5.22) suffices for $r < 2$.

If $p = 0$, then (5.29) follows from (5.3).

To handle $p = 1$, we write

$$(\tau\partial_\tau)\left(ve^{ir\tau}\right) = (-Sv + r(\partial_r + i\tau)v)e^{ir\tau}.$$

The first term is bounded by $|\tau|^{-1}\langle r\rangle^{-1}$ using (5.3), and the second term is bounded by $\langle r\rangle^{-1}$ using (5.22). Both (5.3) and (5.22) hold under our assumptions $p \leq 16p \leq n - 20$ (note the assumption

$16p \le n-20$ implies $p \le n-3$ since p is nonnegative).

For general p, we write

$$(\tau\partial_\tau)^p\left(ve^{ir\tau}\right) = \left(\sum_{j=0}^{p}\sum_{\ell=0}^{j} c_{j\ell} r^\ell (\partial_r + i\tau)^\ell (-S)^{p-j} v\right) e^{ir\tau}.$$

Each term on the right hand side is bounded by

$$|\tau|^{\ell-1}\langle r\rangle^{-1} \lesssim |\tau|^{p-1}\langle r\rangle^{-1}$$

using (5.22) and our assumption $16p \le n-20$. □

Next we find pointwise resolvent bounds for $|\tau| \lesssim 1$. In this case we are interested in the term $R_\tau h_{\nu-3\kappa-3}$ in our expression for $R_\tau g$ when $|\tau| \lesssim 1$ in Proposition 4.5. The terms included in h in the proof of Proposition 4.5 depend on τ, so we consider a τ dependent function g.

Lemma 5.6. *Let $g \in \mathcal{LE}^*$, possibly depending on τ, satisfy*

$$\|\langle r\rangle^q (\partial_r + i\tau)^q T^i \Omega^j S^k g\|_{\mathcal{LE}^*} \lesssim 1, \quad q+i+4j+16k \le n. \tag{5.30}$$

If $\tau \in \mathbb{R}$, $|\tau| \lesssim 1$, and $p < n-3$ then

$$|\partial_r^p (ge^{i\tau r})| \lesssim r^{-p-2}, \tag{5.31}$$

$$|(\partial_r + i\tau)^p g| \lesssim r^{-p-2}, \tag{5.32}$$

and

$$|\partial_r^p g| \lesssim r^{-2}. \tag{5.33}$$

Furthermore, if $|\tau r| \lesssim 1$ then

$$|\partial_r^p g| \lesssim r^{-p-2}. \tag{5.34}$$

Proof. To prove (5.31) we calculate

$$S_r(ge^{i\tau r}) = (r(\partial_r + i\tau)g)e^{i\tau r} \tag{5.35}$$

so that

$$S_r^k(ge^{i\tau r}) = \sum_{m=1}^{k} c_m(r^m(\partial_r + i\tau)^m g)e^{i\tau r}. \tag{5.36}$$

We also have $\Omega^j ge^{i\tau r} = (\Omega^j g)e^{i\tau r}$. Finally we calculate

$$\begin{aligned}|T^i ge^{i\tau r}| &= \Big|\sum_{m=0}^{i} c_m(T^{i-m}g)T^m e^{i\tau r}\Big| \\ &\lesssim \sum_{m=0}^{i} c_m|T^m g|\end{aligned}$$

since we assumed $|\tau| \lesssim 1$. Therefore (5.30) implies $ge^{i\tau r} \in Z^{n,0}$ and (5.31) follows by Lemma 5.2. Then (5.32) follows from (5.31) since

$$\partial_r^p(ge^{i\tau r}) = ((\partial_r + i\tau)^p g)e^{i\tau r}.$$

To prove (5.33), note the case $p = 0$ follows from (5.32). Now assume (5.33) holds for $a < p$. We calculate

$$(\partial_r + i\tau)^p = \sum_{m=0}^{p} c_m\partial_r^m(i\tau)^{p-m} = \partial_r^p + \sum_{m=0}^{p-1} c_m\partial_r^m(i\tau)^{p-m} \tag{5.37}$$

so that

$$|\partial_r^p g| \lesssim |(\partial_r + i\tau)^p g| + \sum_{m=0}^{p-1} |\partial_r^m g|$$

since we assume $|\tau| \lesssim 1$. Then (5.33) follows from (5.32) and the inductive hypothesis.

Finally, to prove (5.34), we see the case $p = 0$ frollows from (5.32). Now assume (5.34) holds for $a < p$. By (5.37) we have

$$|r^p\partial_r^p g| \lesssim |r^p(\partial_r + i\tau)^p g| + \sum_{m=0}^{p-1} r^m|\partial_r^m(i\tau r)^{p-m}| \lesssim \langle r\rangle^{-2}$$

where the last inequality follows from (5.32), the assumption $|\tau r| \lesssim 1$, and the inductive hypothesis.

□

Lemma 5.7. *Let $g \in Z^{n,0}$, possibly depending on τ, satisfy*

$$\|\langle r\rangle^q(\partial_r + i\tau)^q T^i\Omega^j S^k g\|_{\mathcal{LE}^*} \lesssim 1, \quad q + i + 4j + 16k \leq n. \tag{5.38}$$

If $\tau \in \mathbb{R}$ and $|\tau| \lesssim 1$ then $v = R_\tau g$ satisfies

$$|(\partial_r + i\tau)^p v_{ijk}| \lesssim |\tau|^{-1}\langle r\rangle^{-p-1}, \qquad p \le n-3 \quad \text{and } i+4j+16k \le n-20-8p. \tag{5.39}$$

Proof. If $p = 0$, (5.39) follows from (5.4).

To handle $p = 1$, we write

$$(\partial_r^2 + \tau^2)(rv_{ijk}) = -r^{-1}\Delta_\omega v_{ijk} + r(Q_\ell + Q_r)v_{\le i\le j\le k} + rg_{\le i\le j\le k}. \tag{5.40}$$

All but the last term are bounded by $|\tau|^{-1}\langle r\rangle^{-2}$ using (5.4). We note the assumption $g \in Z^{n,0}$ is not enough to allow us to use Lemma 5.2 since g is allowed to depend on τ so $Sg \ne S_r g$. Take $\psi = (\partial_r + i\tau)(rv_{ijk})$. Then the radiation condition (5.5) allows us to integrate from infinity as before to find

$$\begin{aligned}|\psi(r_0)e^{-i\tau r_0}| &= \left|\int_{r_0}^\infty (r^{-1}\Delta_\omega v_{ijk} + r(Q_\ell + Q_r)v_{\le i\le j\le k})e^{-i\tau r}\,dr + \int_{r_0}^\infty rg_{\le i\le j\le k}e^{-i\tau r}\,dr\right| \\ &\lesssim |\tau|^{-1}\langle r_0\rangle^{-1} + \left|\int_{r_0}^\infty rg_{\le i\le j\le k}e^{-i\tau r}\,dr\right|.\end{aligned}$$

For the last term we integrate by parts and use Lemma 5.6 to calculate

$$\begin{aligned}&\left|\int_{r_0}^\infty rg_{\le i\le j\le k}e^{i\tau r}\partial_r\left(\frac{e^{-2i\tau r}}{-2i\tau}\right)\,dr\right| \\ &\quad= \left|\frac{-i}{2\tau}r_0 g_{\le i\le j\le k}(r_0)e^{-i\tau r_0} + \frac{1}{2i\tau}\int_{r_0}^\infty e^{-2i\tau r}\Big(ge^{ir\tau} + r\partial_r(ge^{ir\tau})\Big)\,dr\right| \\ &\quad\lesssim |\tau|^{-1}\langle r_0\rangle^{-1} + |\tau|^{-1}\int_{r_0}^\infty r^{-2}\,dr \\ &\quad\lesssim |\tau|^{-1}\langle r_0\rangle^{-1}.\end{aligned}$$

Thus we have

$$|r(\partial_r + i\tau)v_{ijk}| = |(\partial_r + i\tau)(rv_{ijk}) - v_{ijk}| \lesssim |\tau|\langle r\rangle^{-1}$$

so that $|(\partial_r + i\tau)v_{ijk}| \lesssim |\tau|^{-1}\langle r\rangle^{-2}$, as desired.

We proceed by induction. Fix p and assume

$$|(\partial_r + i\tau)^a v_{ijk}| \lesssim |\tau|^{-1}\langle r\rangle^{-a-1}$$

for $a < p$. We again have (5.25) in which we calculated

$$\begin{aligned}
&(\partial_r - i\tau)(\partial_r + i\tau)^p (r v_{ijk}) \\
&\quad = \sum_{m=0}^{p-1} \Big((-1)^{p-m+1} c_m r^{-(p-m)} (\partial_r + i\tau)^m \Delta_\omega v_{ijk} \Big) + r(\partial_r + i\tau)^{p-1} Q_\ell v_{\leq i \leq j \leq k} \\
&\qquad + C(\partial_r + i\tau)^{p-2} Q_\ell v_{\leq i\leq j\leq k} + r(\partial_r + i\tau)^{p-1} Q_r v_{\leq i\leq j\leq k} + C(\partial_r + i\tau)^{p-2} Q_r v_{ijk} \\
&\qquad + r(\partial_r + i\tau)^{p-1} g_{\leq i\leq j\leq k} + C(\partial_r + i\tau)^{p-2} g_{\leq i\leq j\leq k}.
\end{aligned} \tag{5.41}$$

All but the last two terms are bounded by $|\tau|^{-1}\langle r\rangle^{-p-1}$ using the inductive hypothesis, (5.26), (5.27), and (5.28). For the last two terms we integrate by parts as in the $p = 1$ case to find

$$\begin{aligned}
&\int_{r_0}^{\infty} r(\partial_r + i\tau)^{p-1} g_{\leq i\leq j\leq k} e^{-i\tau r}\, dr \\
&\quad = \frac{i}{2\tau} r_0 (\partial_r + i\tau)^{p-1} g_{\leq i\leq j\leq k}(r_0) e^{-i\tau r_0} + \frac{1}{2i\tau}\int_{r_0}^{\infty} [(\partial_r + i\tau)^{p-1} g + r(\partial_r + i\tau)^p g] e^{-i\tau r}\, dr
\end{aligned}$$

so that by Lemma 5.6 we have

$$\begin{aligned}
\left| \int_{r_0}^{\infty} r(\partial_r + i\tau)^{p-1} g_{\leq i\leq j\leq k} e^{-i\tau r}\, dr \right| &\lesssim |\tau|^{-1}\langle r_0\rangle^{-p} + |\tau|^{-1}\int_{r_0}^{\infty} \langle r\rangle^{-1-p}\, dr \\
&\lesssim |\tau|^{-1}\langle r_0\rangle^{-p}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\int_{r_0}^{\infty} (\partial_r + i\tau)^{p-2} g_{\leq i\leq j\leq k} e^{-i\tau r}\, dr \\
&\quad = \frac{i}{2\tau} (\partial_r + i\tau)^{p-2} g_{\leq i\leq j\leq k}(r_0) e^{-i\tau r_0} + \frac{1}{2i\tau}\int_{r_0}^{\infty} [(\partial_r + i\tau)^{p-1} g_{\leq i\leq j\leq k}] e^{-i\tau r}\, dr
\end{aligned}$$

so that by Lemma 5.6

$$\left| \int_{r_0}^{\infty} (\partial_r + i\tau)^{p} g_{\leq i\leq j\leq k} e^{-i\tau r}\, dr \right| \lesssim |\tau|^{-1}\langle r_0\rangle^{-p}.$$

It follows that

$$|(\partial_r + i\tau)^p v_{ijk}| \lesssim |\tau|^{-1}\langle r\rangle^{-p-1}$$

as desired.

$\square$

Proposition 5.8. *Let $g \in Z^{n,0}$, possibly depending on τ, satisfy*

$$\|\langle r\rangle^q(\partial_r + i\tau)^q T^i \Omega^j S^k g\|_{\mathcal{LE}^*} \lesssim 1, \quad q + i + 4j + 16k \leq n. \tag{5.42}$$

If $\tau \in \mathbb{R}$ and $|\tau| \lesssim 1$ then $v = R_\tau g$ satisfies the following pointwise bounds:

1. *If $\langle r\rangle \lesssim |\tau|^{-1}$, then*

$$|(\tau\partial_\tau)^p v| \lesssim 1, \quad 16p \leq n - 20. \tag{5.43}$$

2. *If $\langle r\rangle \gtrsim |\tau|^{-1}$, then*

$$|(\tau\partial_\tau)^p \left(ve^{i\tau\langle r\rangle}\right)| \lesssim (|\tau|\langle r\rangle)^{-1}, \quad 16p \leq n - 20. \tag{5.44}$$

Proof. **1. Small r:** $\langle r\rangle \lesssim |\tau|^{-1}$

We write

$$\tau\partial_\tau = -S + r\partial_r$$

and find

$$(\tau\partial_\tau)^p v = \sum_{m=0}^{p} c_m (r\partial_r)^{p-m}(-S)^m v.$$

Thus it is sufficient to show $|(r\partial_r)^p v_{00k}| \lesssim 1$ for $16k \leq n - 20 - 16p$. Furthermore, since $(r\partial_r)^p = \sum_{j=0}^{p} c_j r^j \partial_r^j$, it is sufficient to show $|r^p\partial_r^p v_{00k}| \lesssim 1$ for $16k \leq n - 20 - 16p$. As before, we will use (5.40) which introduces Ω and T vector fields, so we will instead bound $|r^p\partial_r^p v_{ijk}|$ then set $i, j = 0$.

When $p = 0$, we have $|v_{ijk}| \lesssim 1$ by (5.4) for $i + 4j + 16k \leq n - 20$. When $p = 1$, we have $|\partial_r v_{ijk}| \lesssim |\nabla v_{ijk}| \lesssim \langle r\rangle^{-1}$ when $i + 4j + 16k \leq n - 20$ by (5.4).

Fix p and assume $|r^a\partial_r^a v_{ijk}| \lesssim 1$ for $a < p$ when $i + 4j + 16k \leq n - 20 - 16a$. Applying $r^{p-2}\partial_r^{p-2}$

to $r^2\partial_r^2 v_{ijk}$ and commuting yields

$$r^p\partial_r^p v_{ijk} = r^{p-2}\partial_r^{p-2}(r^2\partial_r^2 v_{ijk}) - c_1 r^{p-2}\partial_r^{p-2} v_{ijk} - c_2 r^{p-1}\partial_r^{p-1} v_{ijk}. \tag{5.45}$$

The last two terms in (5.45) are bounded by the inductive hypothesis. To handle the first term in (5.45) we use (5.40) and obtain the following expression for $r^2\partial_r^2 v_{ijk}$

$$r^2\partial_r^2 v_{ijk} = -\Delta_\omega v_{ijk} + r^2(Q_\ell + Q_r)v_{\leq i\leq j\leq k} - r^2\tau^2 v_{ijk} + r^2 g_{\leq i\leq j\leq k} - 2r\partial_r v_{ijk}.$$

Now we calculate

$$\begin{aligned}
&|r^{p-2}\partial_r^{p-2}(-\Delta_\omega v_{ijk} + r^2(Q_\ell + Q_r)v_{\leq i\leq j\leq k} - r^2\tau^2 v_{ijk} + r^2 g_{\leq i\leq j\leq k} - 2r\partial_r v_{ijk})| \\
&\quad \lesssim \left|(r^{p-2}\partial_r^{p-4} + c_1 r^{p-1}\partial_r^{p-3} + c_2 r^p\partial_r^{p-2})[(Q_\ell + Q_r)v_{\leq i\leq j\leq k} + g_{\leq i\leq j\leq k} - \tau^2 v_{ijk}]\right| \\
&\qquad + |r^{p-2}\partial_r^{p-2} v_{i(j+2)k}| + |(r^{p-2}\partial_r^{p-2} + r^{p-1}\partial_r^{p-1})v_{ijk}|.
\end{aligned}$$

The Q_ℓ and Q_r terms are handled in a manner analogous to the argument using (5.26), (5.27), and (5.28). Each term on the right hand side is then bounded by the inductive hypothesis, the assumption $|\tau r| \lesssim 1$, and Lemma 5.6.

2. Large r: $\langle r\rangle \gtrsim |\tau|^{-1}$

As in Proposition 5.5, it suffices to prove the proposition for $(\tau\partial_\tau)^k\left(ve^{i\tau r}\right)$.

If $p = 0$, the desired bound follows by (5.4).

Fix p and assume

$$|(\tau\partial_\tau)^a\left(ve^{i\tau r}\right)| \lesssim (|\tau|\langle r\rangle)^{-1}$$

when $a < p$. We write

$$(\tau\partial_\tau)^p(ve^{ir\tau}) = \left(\sum_{j=0}^{p}\sum_{\ell=0}^{j} c_{j\ell} r^\ell(\partial_r + i\tau)^\ell(-S)^{p-j}v\right)e^{ir\tau}.$$

By Lemma 5.7, each term on the right hand side is bounded by

$$|\tau|^{-1}\langle r\rangle^{-1}$$

as desired. $\square$

CHAPTER 6

Proof of main theorem

In this Chapter we prove Theorem 1.4. Recall in (3.21) we stated

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} R_\tau(-i\tau u_0 + P^1 u_0 - u_1) e^{it\tau}\ d\tau.$$

Furthermore we defined the high frequency part of $u(t,x)$, denoted $u_{>1}(t,x)$, by

$$u_{>1}(t,x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \chi_{>1}(|\tau|) R_\tau(-i\tau u_0 + P^1 u_0 - u_1) e^{it\tau}\ d\tau, \tag{6.1}$$

and we defined the low frequency part of $u(t,x)$, denoted $u_{<1}(t,x)$, by

$$u_{<1}(t,x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \chi_{<1}(|\tau|) R_\tau(-i\tau u_0 + P^1 u_0 - u_1) e^{it\tau}\ d\tau \tag{6.2}$$

so $u(t,x) = u_{<1}(t,x) + u_{>1}(t,x)$. We will prove the main theorem for $u_{>1}(t,x)$ and $u_{<1}(t,x)$ separately.

By assumption we have $u_0 \in Z^{\nu+1,\kappa}$ and $u_1 \in Z^{\nu,\kappa+1}$. Since $P^1 : Z^{n,q} \to Z^{n-1,q+\kappa}$, we can write

$$R_\tau(-i\tau u_0 + P^1 u_0 - u_1) = R_\tau(\tau g_\kappa^{\nu+1} + g_{\kappa+1}^{\nu})$$

for some $g_\kappa^{\nu+1} \in Z^{\nu+1,\kappa}$ and some $g_{\kappa+1}^{\nu} \in Z^{\nu,\kappa+1}$. Therefore (6.1) and (6.2) become

$$u_{>1}(t,x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \chi_{>1}(|\tau|) R_\tau(\tau g_\kappa^{\nu+1} + g_{\kappa+1}^{\nu}) e^{i\tau t}\ d\tau \tag{6.3}$$

and

$$u_{<1}(t,x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \chi_{<1}(|\tau|) R_\tau(\tau g_\kappa^{\nu+1} + g_{\kappa+1}^{\nu}) e^{i\tau t}\ d\tau. \tag{6.4}$$

6.1 High Frequency Case ($|\tau| \gtrsim 1$)

In this section we establish pointwise bounds for $u_{>1}(t,x)$. We will decompose the expression $R_\tau(\tau g_\kappa^{\nu+1} + g_{\kappa+1}^\nu)$ in (6.3) via an iterative argument, so we begin by rewriting (6.3) as

$$u_{>1}(t,x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \chi_{>1}(|\tau|) R_\tau(\tau f_0 + g_0) e^{i\tau t}\, d\tau \tag{6.5}$$

for $f_0 = g_\kappa^{\nu+1} \in Z^{\nu+1,\kappa}$ and $g_0 = g_{\kappa+1}^\nu \in Z^{\nu,\kappa+1}$.

We approximate $R_\tau(\tau f_0 + g_0) \approx \tau^{-1} f_0$. A direct calculation shows that the error, denoted u_1, is given by

$$\begin{aligned} u_1 &= R_\tau\left((g_0 - iP^1 f_0) + \tau^{-1}(\Delta + P^2)(-f_0)\right) \\ &=: R_\tau\left(f_1 + \tau^{-1} g_1\right). \end{aligned}$$

Note $P^2 : Z^{p,q} \to Z^{p-2,q+\kappa}$, $P^1 : Z^{p,q} \to Z^{p-1,q+\kappa}$, and $\Delta : Z^{p,q} \to Z^{p-2,q+2}$. The latter can be seen by writing Δ in spherical coordinates and viewing the derivatives as vector fields:

$$\begin{aligned} \Delta &= \partial_r^2 + \frac{2}{r}\partial_r + r^{-2}\Delta_\omega \\ &= r^{-2}(S_r^2 - S_r) + 2r^{-2} S_r + r^{-2}\Omega^2. \end{aligned}$$

Thus $f_1 \in Z^{\nu,\kappa+1}$ and $g_1 \in Z^{\nu-1,\kappa+2}$. Now we have

$$R_\tau(\tau f_0 + g_0) = \tau^{-1} f_0 + R_\tau(f_1 + \tau^{-1} g_1).$$

Next we reiterate the process and approximate $R_\tau(f_1 + \tau^{-1} g_1) \approx \tau^{-2} f_1$. Direct calculation as above yields

$$\begin{aligned} R_\tau(f_1 + \tau^{-1} g_1) &= \tau^{-2} f_1 + R_\tau(\tau^{-1}(g_1 - iP^1 f_1) + \tau^{-2}(\Delta + P^2)(-f_1)) \\ &=: \tau^{-2} f_1 + R_\tau(\tau^{-1} f_2 + \tau^{-2} g_2) \end{aligned}$$

where $f_2 \in Z^{\nu-1,\kappa+2}$ and $g_2 \in Z^{\nu-2,\kappa+3}$. Further reiterating this process a total of J times we obtain the representation

$$R_\tau(\tau f_0 + g_0) = \sum_{j=0}^{J-1} \tau^{-j-1} f_j + \tau^{-J} R_\tau\left(\tau f_J + g_J\right) \tag{6.6}$$

where $f_j = g_{j-1} - iP^1 f_{j-1}$ and $g_j = -(\Delta + P^2) f_{j-1}$. Note that $f_j \in Z^{\nu+1-j,\kappa+j}$ and $g_j \in Z^{\nu-j,\kappa+1+j}$.

Defining

$$\hat{u}_a(\tau) := \sum_{j=0}^{J-1} \tau^{-j-1} f_j$$

and

$$\hat{u}_b(\tau) := \tau^{-J} R_\tau\left(\tau f_J + g_J\right)$$

we see (6.5) and (6.6) yield

$$u_{>1}(t,x) = \int_{\mathbb{R}} \chi_{>1}(|\tau|)(\hat{u}_a(\tau) + \hat{u}_b(\tau)) e^{i\tau t}\, d\tau.$$

We will handle each term separately. Note by Lemma 5.2 $|f_j| \lesssim \langle r\rangle^{-2-\kappa-j}$ since $f_j \in Z^{\nu+1-j,\kappa+j}$. This allows us to calculate

$$\begin{aligned}
\left|\int_{\tau\in\mathbb{R}} \chi_{>1}(|\tau|)\hat{u}_a(\tau) e^{it\tau}\, d\tau\right| &= \left|\sum_{j=0}^{J-1} \int \chi_{>1}(|\tau|)\tau^{-j-1} f_j(x) e^{it\tau}\, d\tau\right| \\
&\lesssim \sum_{j=0}^{J-1} \langle r\rangle^{-\kappa-2-j} \langle t\rangle^{-N} \left|\int \partial_\tau^N (\chi_{>1}(|\tau|)\tau^{-j-1}) e^{i\tau t}\, d\tau\right| \\
&\lesssim \sum_{j=0}^{J-1} \langle r\rangle^{-\kappa-2-j} \langle t\rangle^{-N} \Big(\int_1^\infty |\tau|^{-j-1-N}\, d\tau + \sum_{\ell=0}^{N-1} \int_1^2 |\tau|^{-j-1-\ell}\, d\tau \Big) \\
&\lesssim \langle t\rangle^{-N} \langle r\rangle^{-\kappa-2}
\end{aligned}$$

for any $N \geq 1$.

For $\hat{u}_b$ we use the high frequency resolvent bound (5.29), which yields

$$|(\tau\partial_\tau)^\ell (\hat{u}_b(\tau) e^{i\tau\langle r\rangle})| \lesssim |\tau|^{\ell-J} \langle r\rangle^{-1}$$

for $16\ell \leq \nu - J - 20$ and $\ell \leq \kappa + J$. We calculate

$$
\begin{aligned}
\left|\int_{\tau\in\mathbb{R}} \chi_{>1}(\tau)\hat{u}_b(\tau)e^{it\tau}\ d\tau\right| &= \left|\int \chi_{>1}(\tau)\hat{u}_b e^{ir\tau}e^{i(t-r)\tau}\ d\tau\right| \\
&\approx \langle t-r\rangle^{-N}\left|\int \tau^{-N}\left[\sum_{\ell=0}^{N}(\tau\partial_\tau)^\ell(\chi_{>1}(|\tau|)\hat{u}_b e^{ir\tau})\right]e^{i(t-r)\tau}\ d\tau\right| \\
&\lesssim \langle t-r\rangle^{-N}\int_{|\tau|>1}|\tau|^{-N}\sum_{\ell=0}^{N}|\tau|^{\ell-J}\langle r\rangle^{-1}\ d\tau \\
&= \sum_{\ell=0}^{N}\langle t-r\rangle^{-N}\langle r\rangle^{-1}\int_{|\tau|>1}|\tau|^{-N-J+\ell}\ d\tau \\
&\lesssim \langle t-r\rangle^{-N}\langle r\rangle^{-1}\int_{|\tau|>1}|\tau|^{-J}\ d\tau \\
&\lesssim \langle r\rangle^{-1}\langle t-r\rangle^{-N}
\end{aligned}
$$

for $J \geq 2$, $N \leq \kappa + J$, and $16N \leq \nu - J - 20$.

Combining the above results, we find

$$|u_{>1}(t,x)| \lesssim \langle t\rangle^{-N}\langle r\rangle^{-\kappa-1} + \langle r\rangle^{-1}\langle t-r\rangle^{-N}.$$

Theorem 1.4 then follows in the high frequency case if we take $J = 2$ and $N = \kappa + 2$ since the resulting requirement on ν is $\nu \geq 16\kappa + 53$, which is satisfied by our assumption $\nu \geq 31\kappa + 168$.

6.2 Low Frequency Case ($|\tau| \lesssim 1$)

In this section we establish pointwise bounds for $u_{<1}(t,x)$. We will use Proposition 4.5 to analyze (6.4). Note Proposition 4.3 shows $|R_0 g^\nu_\lambda| \lesssim \langle r\rangle^{-1}$ for $1 \leq \lambda \leq \kappa+1$. Therefore the terms of the form $R_0 g^\nu_\lambda$ in our expressions for $R_\tau g^\nu_\lambda$ in Proposition 4.5 can be included in the terms of the form $F_m(x)$ (as the only assumption on $F_m(x)$ is $|F_m| \lesssim \langle r\rangle^{-1}$). Thus we see

$$
\begin{aligned}
\tau R_\tau g^{\nu+1}_\kappa &= \tau R_\tau(\chi_{>|\tau|^{-1}}(r)g^{\nu+1}_\kappa) + \sum_{m=0}^{\kappa-1}\tau^{m+1}F_m(x)e^{-i\tau\langle r\rangle} + \tau^{\kappa+1}(R_\tau h_{\nu+1-3\kappa}) \\
R_\tau g^\nu_{\kappa+1} &= R_\tau(\chi_{>|\tau|^{-1}}(r)g^\nu_{\kappa+1}) + \sum_{m=0}^{\kappa}\tau^m F_m(x)e^{-i\tau\langle r\rangle} + \tau^\kappa\epsilon(r,\tau)e^{-i\tau\langle r\rangle} + \tau^{\kappa+1}(R_\tau h_{\nu-3\kappa-3})
\end{aligned}
$$

so that

$$\begin{aligned}
&R_\tau(\tau g_{\kappa+1}^{\nu} + g_{\kappa}^{\nu+1}) \\
&\quad = R_\tau\Big(\chi_{>|\tau|^{-1}}(r)(\tau g_\kappa^{\nu+1} + g_{\kappa+1}^{\nu})\Big) + \Big(\sum_{m=0}^{\kappa} \tau^m F_m(x) e^{-i\tau\langle r\rangle}\Big) + \tau^\kappa \epsilon(r,\tau) e^{-i\tau\langle r\rangle} \\
&\qquad + \tau^{\kappa+1}(R_\tau h_{\nu-3\kappa-3})
\end{aligned} \tag{6.7}$$

where

$$\epsilon(r,\tau) = \tau^\kappa \langle r\rangle^{-1}\epsilon_1(r\wedge|\tau|^{-1}) + \tau^{\kappa+1}\big(\epsilon_2(r\wedge|\tau|^{-1}) - \epsilon_2(|\tau|^{-1})\big)$$

with $\epsilon_1, \epsilon_2 \in S_{rad}(\log r)$ and $r\wedge|\tau|^{-1} \approx min(r, |\tau|^{-1})$ is smooth.

Consider the first term on the right hand side of (6.7). Using a decomposition as in (6.6), we find

$$\begin{aligned}
&R_\tau\Big(\chi_{>|\tau|^{-1}}(r)(\tau g_\kappa^{\nu+1} + g_{\kappa+1}^{\nu})\Big) \\
&\qquad = \sum_{j=0}^{J-1} \tau^{-j-1}(\chi_{>|\tau|^{-1}} g_{\kappa+j}^{\nu+1-j}) + \tau^{-J} R_\tau\Big(\tau(\chi_{>|\tau|^{-1}} g_{\kappa+J}^{\nu+1-J}) + \chi_{>|\tau|^{-1}} g_{\kappa+1+J}^{\nu-J}\Big) \\
&\qquad = \sum_{j=0}^{J-1} \tau^{-j-1}(\chi_{>|\tau|^{-1}} g_{\kappa+j}^{\nu+1-j}) + \sum_{M=J-1}^{J} \tau^{-M} R_\tau(\chi_{>|\tau|^{-1}} g_{\kappa+1+M}^{\nu-M}).
\end{aligned} \tag{6.8}$$

We claim

$$\sum_{M=J-1}^{J} \tau^{-M} R_\tau(\chi_{>|\tau|^{-1}} g_{\kappa+1+M}^{\nu-M}) \tag{6.9}$$

can be written as $\tau^{\kappa+1}(R_\tau h_n)$ for $n = \min(\nu-J, J+\kappa)$ for h as in (4.49). In other words, we claim $\tau^{-M-\kappa-1}\chi_{>|\tau|^{-1}} g_{\kappa+1+M}^{\nu-M}$ with $M\in\{J-1, J\}$ satisfies (4.49) for $i+j+k+\ell \le \min(\nu-J, J+\kappa)$. This holds for any J, and we will pick a suitable J once we have determined the necessary regularity for h_n in order for the theorem to hold. To prove the claim, direct calculation yields for any $N\in\mathbb{N}$ and any function g

$$\begin{aligned}
&r^\ell(\partial_r + i\tau)^\ell T^i \Omega^j S^k \tau^{-N}\chi_{>1}(r\tau) g \\
&\quad = \sum_{a=0}^{k}\sum_{b=0}^{i}\sum_{c=0}^{\ell}\sum_{d=0}^{c} (-1)^{k-a} c_a (i\tau)^{\ell-c}\tau^{-N} r^{\ell-c+d}[\partial_r^d T^b \chi_{>1}(r\tau)] r^{c-d}\partial_r^{c-d} T^{i-b}\Omega^j S_r^a g.
\end{aligned}$$

If $b+d \geq 1$ then $|\tau r| \approx 1$ and we find

$$\Big|\sum_{a=0}^{k}\sum_{b=0}^{i}\sum_{c=0}^{\ell}\sum_{d=0}^{c}(-1)^{k-a}c_a(i\tau)^{\ell-c}\tau^{-N}r^{\ell-c+d}[\partial_r^d T^b\chi_{>1}(r\tau)]r^{c-d}\partial_r^{c-d}T^{i-b}\Omega^j S_r^a g\Big|$$
$$\lesssim |\tau|^{\ell-c-N+d+b}|r^{\ell-c+d}T^{\leq i}\Omega^{\leq j}S_r^{\leq k+\ell}g|$$
$$\lesssim |r^N T^{\leq i}\Omega^{\leq j}S_r^{\leq k+\ell}g|$$

since we are in the case $|\tau| \lesssim 1$. Note we have used the fact that in general $r^p\partial_r^p$ can be written as a linear combination of S_r^j for $j \leq p$. If $b+d=0$ then we find

$$\Big|\sum_{a=0}^{k}\sum_{c=0}^{\ell}(-1)^{k-a}c_a(i\tau)^{\ell-c}\tau^{-N}r^{\ell-c}r^c\partial_r^c T^i\Omega^j S_r^a g\Big|$$
$$\lesssim |\tau|^{\ell-c-N}r^{\ell-c}|T^{\leq i}\Omega^{\leq j}S_r^{\leq k+\ell}g|$$
$$\lesssim |r^N T^{\leq i}\Omega^{\leq j}S_r^{\leq k+\ell}g|$$

for $\ell \leq N$ since $|\tau|^{-1} \lesssim r$. It follows that

$$\|r^\ell(\partial_r + i\tau)^\ell T^i\Omega^j S^k \tau^{-\kappa-1-M}\chi_{>|\tau|^{-1}}(r)g_{\kappa+1+M}^{\nu-M}\|_{\mathcal{LE}^*} \lesssim 1$$

for $\ell \leq M+\kappa+1$ and $i+j+k+\ell \leq \nu - M$. This concludes the proof of the claim.

Now using (6.8) and the claim (see (6.9)), (6.7) becomes

$$\begin{aligned} &R_\tau(\tau g_{\kappa+1}^\nu + g_\kappa^{\nu+1}) \\ &\quad= \sum_{j=0}^{J-1}\Big(\tau^{-j-1}(\chi_{>|\tau|^{-1}}g_{\kappa+j}^{\nu+1-j})\Big) + \sum_{m=0}^{\kappa}\Big(\tau^m F_m(x)e^{-i\tau\langle r\rangle}\Big) + \tau^\kappa\epsilon(r,\tau)e^{-i\tau\langle r\rangle} \\ &\qquad + \tau^{\kappa+1}\Big(R_\tau h_{\nu-3\kappa-3} + R_\tau h_n\Big) \end{aligned} \tag{6.10}$$

where J is fixed but arbitrary and $n = \min(\nu - J, J + \kappa)$. We define

$$\begin{aligned}
\hat{u}_a(\tau) &:= \sum_{j=0}^{J-1} \tau^{-j-1}(\chi_{>|\tau|^{-1}}(r) g_{\kappa+j}^{\nu+1-j}) \\
\hat{u}_b(\tau) &:= \sum_{m=0}^{\kappa} \tau^m F_m(x) e^{-i\tau\langle r\rangle} \\
\hat{u}_c(\tau) &:= \tau^\kappa \epsilon(r,\tau) e^{-i\tau\langle r\rangle} \\
\hat{u}_d(\tau) &:= \tau^{\kappa+1}\Big(R_\tau h_{\nu-3\kappa-3} + R_\tau h_n\Big)
\end{aligned}$$

so (6.4) and (6.10) yield

$$u_{<1}(t,x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \chi_{<1}(|\tau|)(\hat{u}_a(\tau) + \hat{u}_b(\tau) + \hat{u}_c(\tau) + \hat{u}_d(\tau)) e^{it\tau}\, d\tau. \tag{6.11}$$

We will bound the terms on the right hand side of (6.11) separately.

.

To handle $\int \chi_{<1}(|\tau|)\hat{u}_a e^{it\tau}\, d\tau$ we calculate

$$\begin{aligned}
\Big|\int_R \chi_{<1}(|\tau|)\hat{u}_a e^{it\tau}\, d\tau\Big| &\le \sum_{j=0}^{J-1} \Big| g_{\kappa+j}^{\nu+1-j} \int_{\mathbb{R}} \chi_{<1}(|\tau|)\chi_{>1}(r|\tau|)\tau^{-j-1} e^{it\tau}\, d\tau\Big| \\
&\lesssim \sum_{j=0}^{J-1} \langle r\rangle^{-\kappa-j-2} t^{-N} \Big| \int_{\mathbb{R}} \partial_\tau^N \Big(\chi_{<1}(|\tau|)\chi_{>1}(r|\tau|)\tau^{-j-1}\Big) e^{it\tau}\, d\tau\Big| \\
&\lesssim \sum_{j=0}^{J-1}\sum_{\ell=0}^{N}\sum_{i=0}^{\ell} \langle r\rangle^{-\kappa-j-2} t^{-N} \Big| \int_{\mathbb{R}} \chi_{<1}^{(N-\ell)}(|\tau|)\chi_{>1}^{(\ell-i)}(r|\tau|) r^{\ell-i}\tau^{-j-i-1} e^{it\tau}\, d\tau\Big|.
\end{aligned} \tag{6.12}$$

Our argument bounding the integral on the right hand side of (6.12) depends on the values of i and ℓ. If $i = \ell = 0$ and $N \ge 1$, then the derivatives landing on $\chi_{<1}(|\tau|)$ bound $|\tau|$ away from zero and we find

$$\Big| \int_{\mathbb{R}} \chi_{<1}^{(N)}(|\tau|)\chi_{>1}(r|\tau|)\tau^{-j-1} e^{it\tau}\, d\tau\Big| \lesssim 1.$$

If $i = \ell \ge 1$, then $|\tau|^{-j-i-1}$ is integrable from infinity for all $j \ge 0$ and we find

$$\Big| \int_{\mathbb{R}} \chi_{<1}^{(N-\ell)}(|\tau|)\chi_{>1}(r|\tau|)\tau^{-j-\ell-1} e^{it\tau}\, d\tau\Big| \lesssim \int_{r^{-1}}^{\infty} |\tau|^{-j-\ell-1}\, d\tau = \langle r\rangle^{j+\ell} \lesssim \langle r\rangle^{j+N}.$$

If $i < \ell$ then the derivatives landing on $\chi_{>1}(r\tau)$ give $|\tau| \approx r^{-1}$ and we find

$$\Big|\int_{\mathbb{R}} \chi_{<1}^{(N-\ell)}(|\tau|)\chi_{>1}^{(\ell-i)}(r|\tau|)r^{\ell-i}\tau^{-j-i-1}e^{it\tau}\ d\tau\Big| \lesssim \langle r\rangle^{\ell+j+1}\int_{|\tau|\approx r^{-1}} 1\ d\tau \lesssim \langle r\rangle^{j+N}.$$

Combining the above then yields

$$\Big|\int_{\mathbb{R}} \chi_{<1}(|\tau|)\hat{u}_a e^{it\tau}\ d\tau\Big| \lesssim \langle r\rangle^{-\kappa-2+N}\langle t\rangle^{-N} \tag{6.13}$$

for $N \geq 1$.

If $r \geq \frac{t}{2}$, we take $N = 1$ in (6.13) and find

$$\Big|\int_{R} \chi_{<1}(|\tau|)\hat{u}_a e^{it\tau}\ d\tau\Big| \lesssim \langle t\rangle^{-1}\langle t-r\rangle^{-\kappa-1} \tag{6.14}$$

since $\langle r\rangle^{-1} \lesssim \langle t-r\rangle^{-1}$.

If $r < \frac{t}{2}$, we take $N = \kappa+2$ in (6.13) and find

$$\Big|\int_{R} \chi_{<1}(|\tau|)\hat{u}_a e^{it\tau}\ d\tau\Big| \lesssim \langle t+r\rangle^{-\kappa-2}. \tag{6.15}$$

Thus by (6.14) and (6.15) we have

$$\Big|\int_{R} \chi_{<1}(|\tau|)\hat{u}_a e^{it\tau}\ d\tau\Big| \lesssim \langle t\rangle^{-1}\langle t-r\rangle^{-\kappa-1}. \tag{6.16}$$

For the $\hat{u}_b$ term in (6.11) we find

$$\begin{aligned}\Big|\int_{\mathbb{R}} \chi_{<1}(|\tau|)\hat{u}_b e^{it\tau}\ d\tau\Big| &\leq \sum_{m=0}^{\kappa}\Big|\int_{\mathbb{R}} \chi_{<1}(|\tau|)\tau^m F_m e^{i(t-\langle r\rangle)\tau}\ d\tau\Big| \\ &\lesssim \frac{1}{\langle t-r\rangle^N}\left|\int \partial_\tau^N\Big(\chi_{<1}(|\tau|)\tau^m F_m(x)\Big)e^{i(t-\langle r\rangle)\tau}\ d\tau\right| \\ &\lesssim \langle r\rangle^{-1}\langle t-r\rangle^{-N}\end{aligned} \tag{6.17}$$

for any N.

Next we handle the $\hat{u}_c$ term in (6.11). Here we have

$$\begin{aligned}\hat{u}_c(\tau) &= \tau^\kappa \epsilon(r,\tau) e^{-i\tau\langle r\rangle} \\ &= \big[\tau^\kappa \langle r\rangle^{-1}\epsilon_1(r\wedge|\tau|^{-1}) + \tau^{\kappa+1}\big(\epsilon_2(r\wedge|\tau|^{-1}) - \epsilon_2(|\tau|^{-1})\big)\big]e^{-i\tau\langle r\rangle}\end{aligned}$$

where $\epsilon_1, \epsilon_2 \in S_{rad}(\log r)$ and $r \wedge |\tau|^{-1} \approx \min(r, |\tau|^{-1})$ is smooth. We define

$$\epsilon_j^m(r) = \int_0^r \beta_{\approx m}(\rho)\partial_\rho \epsilon_j(\rho)\ d\rho$$

so that

$$\epsilon_j(r) = \sum_{m\geq 0} \epsilon_j^m(r)$$

and we have

$$\begin{aligned}&\Big|\int_{\mathbb{R}} \chi_{<1}(|\tau|)\hat{u}_c(\tau)e^{it\tau}\ d\tau\Big| \\ &\quad\leq \sum_{m\geq 0}\Big|\int_{\mathbb{R}} \chi_{<1}(|\tau|)\Big[\langle r\rangle^{-1}\tau^\kappa\epsilon_1^m(r\wedge|\tau|^{-1}) + \tau^{\kappa+1}(\epsilon_2^m(r\wedge|\tau|^{-1}) - \epsilon_2^m(|\tau|^{-1}))\Big]e^{i(t-\langle r\rangle)\tau}\ d\tau\Big|.\end{aligned} \tag{6.18}$$

We note $|\epsilon_j^m(r)| \lesssim 1$ uniformly in m since

$$|\epsilon_j^m(r)| \lesssim \int_0^r \beta_{\approx m}(\rho)\rho^{-1}\ d\rho \lesssim \int_{2^m}^{2^{m+1}} \rho^{-1}\ d\rho = \log 2, \tag{6.19}$$

and for $N \geq 1$ we have the bounds

$$|\partial_r^N \epsilon_j^m(r)| \lesssim 2^{-mN}\mathbf{1}_{\{r\approx 2^m\}}(r). \tag{6.20}$$

Furthermore, we see

$$\epsilon_j^m(r) \equiv \begin{cases} 0, & r \ll 2^m \\ c_m, & r \gg 2^m. \end{cases} \tag{6.21}$$

We integrate the right hand side of (6.18) with r fixed and break the sum into $m \ll \log r$, $m \gg \log r$, and $m \approx \log r$. When $r \gg 2^m$ we have $\epsilon_j^m(r\wedge|\tau|^{-1}) = \epsilon_j^m(|\tau|^{-1})$ since $\epsilon_j^m(|\tau|^{-1}) = \epsilon_j^m(r)$

for $r < |\tau|^{-1}$ so the right hand side of (6.18) yields for $r \gg 2^m$

$$\begin{aligned} &\sum_{m \ll \log r} \Big| \int_{\mathbb{R}} \chi_{<1}(|\tau|) \Big[\langle r \rangle^{-1} \tau^\kappa \epsilon_1^m (r \wedge |\tau|^{-1}) + \tau^{\kappa+1} (\epsilon_2^m (r \wedge |\tau|^{-1}) - \epsilon_2^m (|\tau|^{-1})) \Big] e^{i(t - \langle r \rangle)\tau} \, d\tau \Big| \\ &\qquad = \sum_{m \ll \log r} \Big| \int_{\mathbb{R}} \chi_{<1}(|\tau|) \langle r \rangle^{-1} \tau^\kappa \epsilon_1^m (|\tau|^{-1}) e^{i(t - \langle r \rangle)\tau} \, d\tau \Big| \end{aligned} \tag{6.22}$$

where $\epsilon_1^m(|\tau|^{-1}) = 0$ for $|\tau|^{-1} \ll 2^m$. We change variables by $\tau \to 2^{-m}\tau$ and find

$$\begin{aligned} &\sum_{m \ll \log r} \Big| \int_{\mathbb{R}} \chi_{<1}(|\tau|) \langle r \rangle^{-1} \tau^\kappa \epsilon_1^m (|\tau|^{-1}) e^{i(t - \langle r \rangle)\tau} \, d\tau \Big| \\ &\qquad = \sum_{m \ll \log r} \langle r \rangle^{-1} 2^{-m(\kappa+1)} \Big| \int_{\mathbb{R}} \chi_{<1}(|\tau| 2^{-m}) \tau^\kappa \epsilon_1^m (2^m |\tau|^{-1}) e^{i \frac{(t - \langle r \rangle)\tau}{2^m}} \, d\tau \Big| \\ &\qquad = \sum_{m \ll \log r} \langle r \rangle^{-1} 2^{-m(\kappa+1)} 2^{mN} |t - \langle r \rangle|^{-N} \Big| \int_{\mathbb{R}} \partial_\tau^N \Big(\chi_{<1}(|\tau| 2^{-m}) \tau^\kappa \epsilon_1^m (2^m |\tau|^{-1}) \Big) e^{i \frac{(t - \langle r \rangle)\tau}{2^m}} \, d\tau \Big| \\ &\qquad \lesssim \sum_{m \ll \log r} \langle r \rangle^{-1} 2^{m(N - (\kappa+1))} |t - \langle r \rangle|^{-N} \end{aligned} \tag{6.23}$$

for any N. For the last inequality, we note $\epsilon_1^m(2^m|\tau|^{-1}) = 0$ for $2^{-m}|\tau| \gg 2^{-m}$ by (6.21) so that by (6.19) and (6.20), we see $|\partial_\tau^N \big(\chi_{<1}(|\tau|2^{-m}) \epsilon_1^m(2^m|\tau|^{-1}) \big)| \lesssim 1$ and we are integrating over $|\tau| \lesssim 1$.

To finish out the calculations for (6.18) in the $r \gg 2^m$ case, we consider the sum on the right hand side of (6.23) using different arguments depending on the size of t. If $t < 2r$ then $|t - \langle r \rangle| < r$ and we further break up the sum into $2^m < |t - \langle r \rangle|$ and $2^m \geq |t - \langle r \rangle|$. When $2^m < |t - \langle r \rangle|$ we set $N = \kappa + 2$ to find

$$\begin{aligned} \sum_{m=0}^{\log |t - \langle r \rangle|} \langle r \rangle^{-1} 2^m |t - \langle r \rangle|^{-\kappa-2} &\lesssim \langle r \rangle^{-1} |t - \langle r \rangle|^{-\kappa-1} \\ &\lesssim \langle t + r \rangle^{-1} \langle t - r \rangle^{-\kappa-1} \end{aligned} \tag{6.24}$$

and when $2^m \geq |t - \langle r \rangle|$ we set $N = 0$ to find

$$\begin{aligned} \sum_{m = \log |t - \langle r \rangle|}^{\log r} \langle r \rangle^{-1} 2^{-m(\kappa+1)} &\lesssim \langle r \rangle^{-1} |t - \langle r \rangle|^{-\kappa-1} \\ &\lesssim \langle t + r \rangle^{-1} \langle t - r \rangle^{-\kappa-1} \end{aligned} \tag{6.25}$$

so that by (6.23) we have

$$\sum_{m \ll \log r} \Big| \int_{\mathbb{R}} \chi_{<1}(|\tau|) \langle r \rangle^{-1} \tau^{\kappa} \epsilon_1^m(|\tau|^{-1}) e^{i(t-\langle r \rangle)\tau} \, d\tau \Big| \lesssim \langle t+r \rangle^{-1} \langle t-r \rangle^{-\kappa-1}, \qquad t < 2r. \tag{6.26}$$

If $t \geq 2r$ then $|t - \langle r \rangle|^{-1} \lesssim \langle t+r \rangle^{-1}$ and we set $N = \kappa + 2$ in the right hand side of (6.23) to find

$$\begin{aligned} \sum_{m \ll \log r} \langle r \rangle^{-1} 2^m |t - \langle r \rangle|^{-\kappa-2} &\lesssim |t - \langle r \rangle|^{-\kappa-2} \\ &\lesssim \langle t+r \rangle^{-1} \langle t-r \rangle^{-\kappa-1} \end{aligned} \tag{6.27}$$

so that by (6.23)

$$\sum_{m \ll \log r} \Big| \int_{\mathbb{R}} \chi_{<1}(|\tau|) \langle r \rangle^{-1} \tau^{\kappa} \epsilon_1^m(|\tau|^{-1}) e^{i(t-\langle r \rangle)\tau} \, d\tau \Big| \lesssim \langle t+r \rangle^{-1} \langle t-r \rangle^{-\kappa-1}, \qquad t \geq 2r. \tag{6.28}$$

Combining (6.22), (6.26), and (6.28) then yields

$$\begin{aligned} \sum_{m \ll \log r} \Big| \int_{\mathbb{R}} \chi_{<1}(|\tau|) \Big[\langle r \rangle^{-1} \tau^{\kappa} \epsilon_1^m(r \wedge |\tau|^{-1}) + \tau^{\kappa+1} (\epsilon_2^m(r \wedge |\tau|^{-1}) - \epsilon_2^m(|\tau|^{-1})) \Big] e^{i(t-\langle r \rangle)\tau} \, d\tau \Big| \\ \lesssim \langle t+r \rangle^{-1} \langle t-r \rangle^{-\kappa-1}. \end{aligned} \tag{6.29}$$

We continue analyzing the right hand side of (6.18) now considering the case $2^m \gg r$. When $2^m \gg r$ we have $\epsilon_j^m(r \wedge |\tau|^{-1}) = 0$ for all τ since $r \wedge |\tau|^{-1} < r \ll 2^m$ for all τ so the right hand side of (6.18) yields for $2^m \gg r$

$$\begin{aligned} &\sum_{m \gg \log r} \Big| \int_{\mathbb{R}} \chi_{<1}(|\tau|) \Big[\langle r \rangle^{-1} \tau^{\kappa} \epsilon_1^m(r \wedge |\tau|^{-1}) + \tau^{\kappa+1} (\epsilon_2^m(r \wedge |\tau|^{-1}) - \epsilon_2^m(|\tau|^{-1})) \Big] e^{i(t-\langle r \rangle)\tau} \, d\tau \Big| \\ &\qquad = \sum_{m \gg \log r} \Big| \int_{\mathbb{R}} \chi_{<1}(|\tau|) \tau^{\kappa+1} \epsilon_2^m(|\tau|^{-1}) e^{i(t-\langle r \rangle)\tau} \, d\tau \Big| \end{aligned} \tag{6.30}$$

where $\epsilon_2^m(|\tau|^{-1}) = 0$ for $|\tau|^{-1} \ll 2^m$. We change variables by $\tau \to 2^{-m}\tau$ and find

$$\begin{aligned}
&\sum_{m \gg \log r} \Big| \int_{\mathbb{R}} \chi_{<1}(|\tau|)\tau^{\kappa+1}\epsilon_2^m(|\tau|^{-1})e^{i(t-\langle r\rangle)\tau}\ d\tau \Big| \\
&\qquad = \sum_{m \gg \log r} 2^{-m(\kappa+2)} \Big| \int_{\mathbb{R}} \chi_{<1}(|\tau| 2^{-m})\epsilon_2^m(2^m|\tau|^{-1})e^{i(t-\langle r\rangle)\tau 2^{-m}}\ d\tau \Big| \\
&\qquad = \sum_{m \gg \log r} 2^{-m(\kappa+2)} 2^{mN} |t-\langle r\rangle|^{-N} \Big| \int_{\mathbb{R}} \Big(\partial_\tau^N \chi_{<1}(|\tau| 2^{-m})\epsilon_2^m(2^m|\tau|^{-1})\Big)e^{i(t-\langle r\rangle)\tau 2^{-m}}\ d\tau \Big| \\
&\qquad \lesssim \sum_{m \gg \log r} 2^{m(N-(\kappa+2))} |t-\langle r\rangle|^{-N}.
\end{aligned} \tag{6.31}$$

The last inequality in (6.31) follows by the same argument used for the last inequality in (6.23) this time for $\epsilon_2^m(|\tau|^{-1})$.

To finish out the calculations for (6.18) in the $r \ll 2^m$ case we consider the sum on the right hand side of (6.31) using different arguments depending on the size of t. If $t \leq 2r$ then $\langle r\rangle^{-1} \lesssim \langle t+r\rangle^{-1}$ and $\langle r\rangle^{-1} \lesssim \langle t-r\rangle^{-1}$. We set $N = 0$ in the right hand side of (6.31) to find

$$\sum_{m \gg \log r} 2^{-m(\kappa+2)} \lesssim \langle r\rangle^{-\kappa-2} \lesssim \langle t+r\rangle^{-1}\langle t-r\rangle^{-\kappa-1}$$

so that by (6.31) we have

$$\sum_{m \gg \log r} \Big| \int_{\mathbb{R}} \chi_{<1}(|\tau|)\tau^{\kappa+1}\epsilon_2^m(|\tau|^{-1})e^{i(t-\langle r\rangle)\tau}\ d\tau \Big| \lesssim \langle t+r\rangle^{-1}\langle t-r\rangle^{-\kappa-1}, \qquad t \leq 2r. \tag{6.32}$$

If $t > 2r$ then $r \lesssim |t-\langle r\rangle|$ and $t \lesssim |t-\langle r\rangle|$ so $|t-\langle r\rangle|^{-1} \lesssim \langle t+r\rangle^{-1}$. Here we further break up the sum into $2^m < |t-\langle r\rangle|$ and $2^m \geq |t-\langle r\rangle|$. When $2^m < |t-\langle r\rangle|$ we set $N = \kappa+3$ in the right hand side of (6.31) to find

$$\begin{aligned}
\sum_{m \gg \log r}^{\log|t-\langle r\rangle|} 2^m |t-\langle r\rangle|^{-\kappa-3} &\lesssim |t-\langle r\rangle|^{-\kappa-2} \\
&\lesssim \langle t+r\rangle^{-1}\langle t-r\rangle^{-\kappa-1},
\end{aligned} \tag{6.33}$$

and when $2^m \geq |t - \langle r\rangle|$ we set $N = 0$ in the right hand side of (6.31) to find

$$\begin{aligned} \sum_{m=\log|t-\langle r\rangle|}^{\infty} 2^{-(\kappa+2)m} &\lesssim |t-\langle r\rangle|^{-\kappa-2} \\ &\lesssim \langle t+r\rangle^{-1}\langle t-r\rangle^{-\kappa-1} \end{aligned} \tag{6.34}$$

so that by (6.31) we have

$$\sum_{m\gg\log r} \Big| \int_{\mathbb{R}} \chi_{<1}(|\tau|)\tau^{\kappa+1}\epsilon_2^m(|\tau|^{-1})e^{i(t-\langle r\rangle)\tau}\, d\tau \Big| \lesssim \langle t+r\rangle^{-1}\langle t-r\rangle^{-\kappa-1}, \qquad t > 2r. \tag{6.35}$$

Combining (6.22), (6.26), and (6.28) then yields

$$\begin{aligned} \sum_{m\gg\log r} \Big| \int_{\mathbb{R}} \chi_{<1}(|\tau|)\Big[\tau^\kappa \epsilon_1^m(r\wedge|\tau|^{-1}) + \tau^{\kappa+1}(\epsilon_2^m(r\wedge|\tau|^{-1}) - \epsilon_2^m(|\tau|^{-1}))\Big]e^{i(t-\langle r\rangle)\tau}\, d\tau\Big| \\ \lesssim \langle t+r\rangle^{-1}\langle t-r\rangle^{-\kappa-1}. \end{aligned} \tag{6.36}$$

Finally we consider (6.18) for $2^m \approx r$. Here the summation in m is finite. We find

$$\epsilon_1^m(r\wedge|\tau|^{-1}) = \begin{cases} \epsilon_1^m(|\tau|^{-1}), & |\tau|^{-1} < r \approx 2^m \\ \epsilon_1(r), & |\tau|^{-1} > r \approx 2^m \end{cases} \tag{6.37}$$

and

$$\epsilon_2^m(r\wedge|\tau|^{-1}) - \epsilon_2^m(|\tau|^{-1}) = \begin{cases} 0, & |\tau|^{-1} < r \approx 2^m \\ \epsilon_2^m(r) - \epsilon_2^m(|\tau|^{-1}), & |\tau|^{-1} > r \approx 2^m. \end{cases} \tag{6.38}$$

By (6.37) and (6.38) when $2^m \approx r$ we see (6.18) has terms as in the $2^m \ll r$ and $2^m \gg r$. Thus we argue as above with the added benefit that the summation is finite in m to find

$$\begin{aligned} \sum_{m\approx\log r} \Big| \int_{\mathbb{R}} \chi_{<1}(|\tau|)\Big[\tau^\kappa \epsilon_1^m(r\wedge|\tau|^{-1}) + \tau^{\kappa+1}(\epsilon_2^m(r\wedge|\tau|^{-1}) - \epsilon_2^m(|\tau|^{-1}))\Big]e^{i(t-\langle r\rangle)\tau}\, d\tau\Big| \\ \lesssim \langle t+r\rangle^{-1}\langle t-r\rangle^{-\kappa-1}. \end{aligned} \tag{6.39}$$

Now (6.18), (6.29), (6.36), and (6.39) yield

$$\Big|\int \chi_{<1}(|\tau|)\hat{u}_c(\tau)e^{it\tau}\ d\tau\Big| \lesssim \langle t+r\rangle^{-1}\langle t-r\rangle^{-\kappa-1}. \tag{6.40}$$

Finally we handle the $\hat{u}_d$ term in (6.11). We first do our calculations for h_n then apply the results to $h_{\nu-3\kappa-3}$. Thus we wish to obtain bounds on

$$\int_{\mathbb{R}} \chi_{<1}(|\tau|)\tau^{\kappa+1}(R_\tau h_n)e^{it\tau}\ d\tau.$$

By Proposition 5.8 we have

$$\Big|(\tau\partial_\tau)^\ell(\tau^{\kappa+1}(R_\tau h_n))\Big| \lesssim \tau^{\kappa+1}, \qquad \langle r\rangle \lesssim |\tau|^{-1}, \quad 16\ell \leq n-20 \tag{6.41}$$

and

$$\Big|(\tau\partial_\tau)^\ell\left(\tau^{\kappa+1}(R_\tau h_n)e^{i\tau\langle r\rangle}\right)\Big| \lesssim |\tau|^\kappa\langle r\rangle^{-1}, \qquad \langle r\rangle \gtrsim |\tau|^{-1}, \quad 16\ell \leq n-20. \tag{6.42}$$

We note since $\tau^\ell\partial_\tau^\ell$ can be written as a linear combination of $(\tau\partial_\tau)^a$ with $1 \leq a \leq \ell$, we see in general for any ψ that $|\tau^\ell\partial_\tau^\ell\psi| \lesssim \sum_{a=1}^{\ell}|(\tau\partial_\tau)^a\psi|$. Thus (6.41) and (6.42) hold for $(\tau\partial_\tau)^\ell$ replaced by $\tau^\ell\partial_\tau^\ell$.

We split up the $\langle r\rangle \lesssim |\tau|^{-1}$ and $\langle r\rangle \gtrsim |\tau|^{-1}$ cases using cutoff functions by writing

$$\begin{aligned}
&\Big|\int \chi_{<1}(|\tau|)\tau^{\kappa+1}(R_\tau h_n)e^{it\tau}\ d\tau\Big| \\
&\quad = \Big|\int \chi_{<1}(|\tau|)\Big(\chi_{<1}(r|\tau|)+\chi_{>1}(r|\tau|)\Big)\tau^{\kappa+1}(R_\tau h_n)e^{it\tau}\ d\tau\Big| \\
&\quad \lesssim \Big|\int \chi_{<1}(\langle r\rangle|\tau|)\tau^{\kappa+1}(R_\tau h_n)e^{it\tau}\ d\tau\Big| + \Big|\int \chi_{<1}(|\tau|)\chi_{>1}(\langle r\rangle|\tau|)\tau^{\kappa+1}(R_\tau h_n)e^{i\langle r\rangle\tau}e^{i(t-\langle r\rangle)\tau}\ d\tau\Big|.
\end{aligned} \tag{6.43}$$

To handle the first term in (6.43), we begin by considering the case where $t \leq 2r$. Here we have

$$\begin{aligned}
\Big|\int \chi_{<1}(\langle r\rangle|\tau|)\tau^{\kappa+1}(R_\tau h_n)e^{it\tau}\ d\tau\Big| &\lesssim \int \chi_{<1}(\langle r\rangle|\tau|)|\tau|^{\kappa+1}\ d\tau \\
&\lesssim \langle r\rangle^{-\kappa-2} \\
&\lesssim \langle t\rangle^{-\kappa-2}
\end{aligned}$$

by (6.41). Continuing with the first term in (6.43), we now consider the case where $2r < t$. We calculate

$$\begin{aligned}\left|\int \chi_{<1}(\langle r\rangle|\tau|)\tau^{\kappa+1}(R_\tau h_n)e^{it\tau}\ d\tau\right| &\leq \left|\int_0^{\frac{1}{t}} |\tau|^{\kappa+1}\ d\tau\right| + \left|\int_{\frac{1}{t}}^{\infty} \chi_{<1}(r|\tau|)\tau^{\kappa+1}R_\tau h e^{it\tau}\ d\tau\right| \\ &\lesssim t^{-\kappa-2} + \left|\int_{\frac{1}{t}}^{\infty} \chi_{<1}(r|\tau|)\tau^{\kappa+1}(R_\tau h_n)e^{it\tau}\ d\tau\right|.\end{aligned}$$

We will handle the remaining integral using integration by parts. Define

$$\phi_1(\tau) := \chi_{<1}(\langle r\rangle|\tau|)\tau^{\kappa+1}(R_\tau h_n).$$

By (6.41) we have $|\partial_\tau^N \phi_1(\tau)| \lesssim |\tau|^{\kappa+1-N}$ for $16N \leq n-20$. Therefore integrating by parts $\kappa+4$ times yields

$$\begin{aligned}\left|\int_{\frac{1}{t}}^{\infty} \chi_{<1}(r|\tau|)\tau^{\kappa+1}(R_\tau h_n)e^{it\tau}\ d\tau\right| &= \left|\int_{\frac{1}{t}}^{\infty} \phi_1 e^{it\tau}\ d\tau\right| \\ &\lesssim \sum_{\ell=0}^{\kappa+3} \frac{1}{t^{\ell+1}}|\partial_\tau^\ell \phi_1(\frac{1}{t})| + \frac{1}{t^{\kappa+4}}\left|\int_{\frac{1}{t}}^{\infty} \partial_\tau^{\kappa+4}\phi_1(\tau)e^{it\tau}\ d\tau\right| \\ &\lesssim \sum_{\ell=0}^{\kappa+3} t^{-\ell-1}t^{\ell-\kappa-1} + t^{-\kappa-4}\int_{\frac{1}{t}}^{\infty} |\tau|^{-3}\ d\tau \\ &\lesssim t^{-\kappa-2}\end{aligned} \tag{6.44}$$

for $16(\kappa+4) < n-20$.

Now we consider the second term in (6.43). Here we define

$$\phi_2(x,\tau) := \chi_{<1}(|\tau|)\chi_{>1}(\langle r\rangle|\tau|)\tau^{\kappa+1}(R_\tau h_n)e^{i\tau r}$$

and use (6.42) to calculate

$$
\begin{aligned}
|\partial_\tau^N \phi_2| &\lesssim \sum_{\ell=0}^{N}\sum_{i=0}^{\ell}\Big|\chi_{<1}^{(N-\ell)}(|\tau|)\chi_{>1}^{(\ell-i)}(\langle r\rangle|\tau|)\langle r\rangle^{\ell-i}\Big(\partial_\tau^i \tau^{\kappa+1}(R_\tau h_n)e^{i\langle r\rangle\tau}\Big)\Big| \\
&\lesssim \sum_{\ell=0}^{N}\Big(|\tau|^{\kappa-\ell}\langle r\rangle^{-1} + \sum_{i=0}^{\ell-1}\chi_{>1}^{(\ell-i)}(\langle r\rangle|\tau|)\langle r\rangle^{\ell-1-i}|\tau|^{\kappa-i}\Big) \\
&\lesssim |\tau|^{\kappa-N}\langle r\rangle^{-1} + \sum_{i=0}^{N-1}\chi_{>1}^{(N-i)}(\langle r\rangle|\tau|)\langle r\rangle^{N-\kappa-1}
\end{aligned}
\tag{6.45}
$$

for $16N \leq n-20$, since $\langle r\rangle|\tau| \approx 1$ on the support of $\chi_{>1}^{(j)}(\langle r\rangle|\tau|)$ for $j \geq 1$. Our argument differs depending on the size of t. If $2r \leq t$, then $|t-\langle r\rangle|^{-1} \lesssim \langle t+r\rangle^{-1}$. We use (6.45) to find for $N \geq \kappa+2$ and $16N \leq n-20$

$$
\begin{aligned}
&\Big|\int \chi_{<1}(|\tau|)\chi_{>1}(\langle r\rangle|\tau|)\tau^{\kappa+1}(R_\tau h_n)e^{i\langle r\rangle\tau}e^{i(t-\langle r\rangle)\tau}\ d\tau\Big| \\
&\quad= \Big|\int \phi_2(\tau)e^{i(t-\langle r\rangle)\tau}\ d\tau\Big| \\
&\quad= \langle t-r\rangle^{-N}\Big|\int (\partial_\tau^N\phi_2)e^{i(t-\langle r\rangle)\tau}\ d\tau\Big| \\
&\quad= t^{-1}\langle t-r\rangle^{-N}\Big|\int \partial_\tau\Big[(\partial_\tau^N\phi_2)e^{-i\langle r\rangle\tau}\Big]e^{it\tau}\ d\tau\Big| \\
&\quad\lesssim t^{-1}\langle t-r\rangle^{-N}\Big(\int_{r^{-1}<|\tau|<1}|\partial_\tau^{N+1}\phi_2|\ d\tau + \langle r\rangle\int_{\mathbb{R}}|\partial_\tau^N\phi_2|\ d\tau\Big) \\
&\quad\lesssim t^{-1}\langle t-r\rangle^{-N}\Big(\langle r\rangle^{-1}\int_{\langle r\rangle^{-1}}^{\infty}|\tau|^{\kappa-N-1}\ d\tau + \int_{|\tau|\approx\langle r\rangle^{-1}}\langle r\rangle^{N-\kappa} + \int_{\langle r\rangle^{-1}}^{\infty}|\tau|^{\kappa-N}\ d\tau\Big) \\
&\quad\lesssim t^{-1}\langle t-r\rangle^{-N}\langle r\rangle^{N-\kappa-1} \\
&\quad\lesssim \frac{1}{t\langle t-r\rangle^{\kappa+1}}\frac{\langle r\rangle^{N-\kappa-1}}{\langle t+r\rangle^{N-\kappa-1}}
\end{aligned}
\tag{6.46}
$$

when $t \geq 2r$.

If $t < 2r$ then $1 \lesssim \frac{\langle r\rangle}{\langle t+r\rangle}$, $\langle r\rangle^{-1} \lesssim t^{-1}$, and $\langle r\rangle^{-1} < |t-\langle r\rangle|^{-1} < \infty$. We write

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}}\chi_{<1}(|\tau|)\chi_{>1}(\langle r\rangle|\tau|)\tau^{\kappa+1}(R_\tau h_n)e^{i\langle r\rangle\tau}e^{i(t-\langle r\rangle)\tau}\ d\tau\Big| \\
&\quad\leq \int_0^{|t-\langle r\rangle|^{-1}}\tau^\kappa\langle r\rangle^{-1}\ d\tau + \Big|\int_{|t-\langle r\rangle|^{-1}}^{\infty}\phi_2(\tau)e^{i(t-\langle r\rangle)\tau}\ d\tau\Big| \\
&\quad\lesssim t^{-1}|t-\langle r\rangle|^{-\kappa-1} + \Big|\int_{|t-\langle r\rangle|^{-1}}^{\infty}\phi_2(\tau)e^{i(t-\langle r\rangle)\tau}\ d\tau\Big|.
\end{aligned}
\tag{6.47}
$$

For the second term in (6.47) we argue as in (6.44) using (6.45) to bound $|\partial_\tau^N \phi_2|$ and obtain

$$\Big| \int_{|t-\langle r\rangle|^{-1}}^{\infty} \phi_2(\tau) e^{i(t-\langle r\rangle)\tau} \, d\tau \Big| \lesssim \langle r\rangle^{-1} |t-\langle r\rangle|^{-\kappa-1} \lesssim t^{-1}\langle t-r\rangle^{-\kappa-1}, \tag{6.48}$$

which combined with (6.47) yields

$$\begin{aligned} &\Big| \int_{\mathbb{R}} \chi_{<1}(|\tau|)\chi_{>1}(\langle r\rangle|\tau|)\tau^{\kappa+1}(R_\tau h_n) e^{i\langle r\rangle\tau} e^{i(t-\langle r\rangle)\tau} \, d\tau \Big| \\ &\qquad \lesssim \frac{1}{t\langle t-r\rangle^{\kappa+1}} \frac{\langle r\rangle^{N-\kappa-1}}{\langle t+r\rangle^{N-\kappa-1}} \end{aligned} \tag{6.49}$$

for $2r > t$ and $16(\kappa+4) \leq n-20$ since in this region $1 \lesssim \frac{\langle r\rangle}{\langle t+r\rangle}$.

Combining (6.43), (6.44), (6.46), and (6.49) then yields

$$\Big| \int_{\mathbb{R}} \chi_{<1}(|\tau|)\tau^{\kappa+1}(R_\tau h_n) e^{it\tau} \, d\tau \Big| \lesssim \frac{1}{\langle t+r\rangle^{\kappa+2}} + \frac{1}{t\langle t-r\rangle^{\kappa+1}} \frac{\langle r\rangle^{N-\kappa-1}}{\langle t+r\rangle^{N-\kappa-1}} \tag{6.50}$$

for $16(\kappa+4) \leq n-20$ and $16N \leq n-20$. We note the statement of the main theorem holds for $N \geq \kappa+1$. If we take $J = 15\kappa+84$, then $16(\kappa+4) \leq J+\kappa-20$ and our assumption $\nu \geq 31\kappa+168$ gives $16(\kappa+4) \leq \nu - J - 20$. Thus (6.50) holds for $n = \min(\nu - J, J+\kappa)$. Similarly, our assumption on ν guarantees (6.50) holds for $n = \nu - 3\kappa - 3$.

The statement of the main theorem then follows for $|\tau| \lesssim 1$ by (6.11), (6.16), (6.17), (6.40), and (6.50).

APPENDIX A

DETAILED CALCULATIONS

A.1 Some Lemmas

A.1.1 Function Spaces

Lemma A.1. *Let $\rho_\ell^q \in \ell^1 S(r^q)$ and $\rho_r^q \in S_{rad}(r^q)$ be given. Then for any $\phi \in \mathcal{LE}$,*

$$\|\rho_\ell^q \phi\|_{\mathcal{LE}^*} \lesssim \|\phi\|_{\mathcal{LE}}, \quad q \leq -1$$

and

$$\|\rho_r^q \phi\|_{\mathcal{LE}^*} \lesssim \|\phi\|_{\mathcal{LE}}, \quad q < -1.$$

Proof. The lemma is proved by direct calculation. We find

$$\begin{aligned}
\|\rho_\ell^q \phi\|_{\mathcal{LE}^*} &= \sum_m \|\langle r\rangle^{\frac{1}{2}} \rho_\ell^q \phi\|_{L^2(A_m)} \\
&= \sum_m \|\langle r\rangle^{-\frac{1}{2}} \langle r\rangle \rho_\ell^q \phi\|_{L^2(A_m)} \\
&\lesssim \sum_m \sup_{A_m} \langle r\rangle \rho_\ell^q \|\langle r\rangle^{-\frac{1}{2}} \phi\|_{L^2(A_m)} \\
&\lesssim \|\phi\|_{\mathcal{LE}} \sum_m \sup_{A_m} \langle r\rangle \rho_\ell^q.
\end{aligned}$$

Since $\sum_m \sup_{A_m} \langle r\rangle \rho_\ell^q \lesssim 1$ for $q \leq -1$, the first part of the proof is concluded.

Similarly we find

$$\|\rho_r^q \phi\|_{\mathcal{LE}^*} \lesssim \|\phi\|_{\mathcal{LE}} \sum_m \sup_{A_m} \langle r\rangle \rho_r^q.$$

In this case $\sum_m \sup_{A_m} \langle r\rangle \rho_r^q \lesssim 1$ for $q < -1$, concluding the second part of the proof.

□

Lemma A.2. *If $R \geq 1$, then*

$$\|\langle r\rangle^q \chi_{>R} f\|_{\mathcal{LE}^*} \lesssim R^{-1} \|f\|_{L^\infty}$$

for $q \leq -3$.

Proof. We calculate

$$\begin{aligned}
\|\langle r\rangle^q \chi_{>R} f\|_{\mathcal{LE}^*} &\leq \|f\|_{L^\infty} \sum_{m>\log R} \|\langle r\rangle^{q+\frac{1}{2}}\|_{L^2(A_m)} \\
&\lesssim \|f\|_{L^\infty} \sum_{m>\log R} 2^{m(q+2)} \\
&\approx \|f\|_{L^\infty} R^{q+2} \sum_m 2^{m(q+2)}.
\end{aligned}$$

Thus the desired inequality holds for $q \leq -3$ when $R \geq 1$. ☐

Lemma A.3. *Let $\rho^q \in S(r^q)$. If $\phi \in \ell^1 S(1)$, then*

$$\|(\partial^\alpha \phi)\rho^q\|_{Z^{n,\lambda}} \lesssim \sum_{|B|=|\alpha|}^{n+|\alpha|} \sum_m \|\langle r\rangle^{|B|} \partial^B \phi\|_{L^\infty(A_m)} \tag{A.1}$$

for $q \leq -\lambda - 2 + |\alpha|$.

Proof. Since $\phi \in \ell^1 S(1)$, note the right hand side of (A.1) is bounded by assumption. We calculate

$$\begin{aligned}
\|(\partial^\alpha \phi)\rho^q\|_{Z^{n,\lambda}} &\lesssim \sup_{i+j+k\leq n} \sum_m 2^{m(\frac{1}{2}+\lambda)} \|T^i \Omega^j S_r^k [(\partial^\alpha \phi)\rho^q]\|_{L^2(A_m)} \\
&\lesssim \sum_m \sum_{|\beta|=0}^{n} 2^{m(\frac{1}{2}+\lambda)} \|\langle r\rangle^{|\beta|} \partial^\beta [(\partial^\alpha \phi)\rho^q]\|_{L^2(A_m)} \\
&\lesssim \sum_m \sum_{|\beta|=0}^{n} \sum_{|J|+|K|=|\beta|} 2^{m(\frac{1}{2}+\lambda+|\beta|)} \|(\partial^J \partial^\alpha \phi)\partial^K \rho^q\|_{L^2(A_m)} \\
&\lesssim \sum_m \sum_{|\beta|=0}^{n} \sum_{|J|+|K|=|\beta|} 2^{m(\frac{1}{2}+\lambda+|\beta|)} \|\rho^{q-|\beta|-|\alpha|} \rho^{|\beta|-|K|+|\alpha|} (\partial^J \partial^\alpha \phi)\|_{L^2(A_m)} \\
&\lesssim \sum_m \sum_{|\beta|=0}^{n} \sum_{|J|=0}^{|\beta|} 2^{m(2+\lambda+q-|\alpha|)} \|\rho^{|J|+|\alpha|} (\partial^J \partial^\alpha \phi)\|_{L^\infty(A_m)} \\
&\lesssim \sum_m \sum_{|B|=|\alpha|}^{|\alpha|+n} 2^{m(2+\lambda+q-|\alpha|)} \|\rho^{|B|} (\partial^B \phi)\|_{L^\infty(A_m)}.
\end{aligned}$$

If $q \leq -\lambda - 2 + |\alpha|$, then $2 + \lambda + q - |\alpha| \leq 0$ and we have

$$\sum_m \sum_{|B|=|\alpha|}^{|\alpha|+n} 2^{m(2+\lambda+q-|\alpha|)} \|\rho^{|B|} (\partial^B \phi)\|_{L^\infty(A_m)} \lesssim \sum_{|B|=|\alpha|}^{|\alpha|+n} \sum_m \|\langle r\rangle^{|B|} (\partial^B \phi)\|_{L^\infty(A_m)}$$

as desired. ∎

Lemma A.4. *If $\phi \in Z^{n,\lambda}$, then*

$$\|\nabla\phi\|_{Z^{n-1,\lambda+1}} \lesssim \|\phi\|_{Z^{n,\lambda}}.$$

Proof. Using $|\nabla\phi| \lesssim |r^{-1}S_r\phi| + |r^{-1}\Omega\phi|$ and $[S_r, \Omega] = 0$, we find

$$\begin{aligned}
\|\nabla\phi\|_{Z^{n-1,\lambda+1}} &\lesssim \sup_{i+j+k\le n-1} \|\langle r\rangle^{\lambda+1} T^i \Omega^j S_r^k r^{-1}\Omega q\|_{\mathcal{LE}^*} + \|\langle r\rangle^{\lambda+1} T^i \Omega^j S_r^k r^{-1} S_r q\|_{\mathcal{LE}^*} \\
&\lesssim \sup_{i+j+k\le n-1} \|\langle r\rangle^{\lambda} T^i \Omega^{j+1} S_r^k q\|_{\mathcal{LE}^*} + \|\langle r\rangle^{\lambda+1} [T^i \Omega^j S_r^k, r^{-1}]\Omega q\|_{\mathcal{LE}^*} \\
&\qquad \|\langle r\rangle^{\lambda} T^i \Omega^j S_r^{k+1} q\|_{\mathcal{LE}^*} + \|\langle r\rangle^{\lambda+1} [T^i \Omega^j S_r^k, r^{-1}] S_r q\|_{\mathcal{LE}^*}.
\end{aligned}$$

For the commutators we find

$$[S_r, r^{-1}] = -r^{-1}, \qquad [\Omega, r^{-1}] = 0,$$

and if $\rho_\ell^{-1} \in \ell^1 S(r^{-1})$, then

$$[T, \rho_\ell^{-1}] \in \ell^1 S(r^{-2}).$$

Therefore we have

$$\|\nabla\phi\|_{Z^{n-1,\lambda+1}} \lesssim \|\phi\|_{Z^{n,\lambda}}$$

as desired. ∎

A.1.2 Commutators

Lemma A.5. *If $r \gtrsim 1$ then*

$$[\frac{x}{r}, \Delta] = -2\frac{1}{r}\nabla + 2\frac{1}{r}\frac{x}{r}(\partial_r + \frac{1}{r})$$

Proof.

$$\begin{aligned}
\Delta\Big[\Big(\frac{x_j}{r}\Big)f\Big] &= \frac{x_j}{r}\Delta f + 2\nabla\Big(\frac{x_j}{r}\Big)\cdot\nabla f + \Delta\Big[\Big(\frac{x_j}{r}\Big)f\Big] \\
&= \frac{x_j}{r}\Delta f + 2\Big(\frac{\delta_{ij}}{r} - \frac{x_i x_j}{r^3}\Big)\partial_i f + 2\nabla x_j \cdot \nabla r^{-1} f \\
&= \frac{x_j}{r}\Delta f + \frac{2}{r}\partial_j f - 2\frac{x_j}{r^2}\partial_r f - 2\frac{x_j}{r^3} f \\
&= \frac{x_j}{r}\Delta f + 2\frac{1}{r}\partial_j f - 2\frac{x_j}{r^2}(\partial_r + \frac{1}{r})f.
\end{aligned}$$

$\square$

Lemma A.6.

$$[(\partial_r + \frac{1}{r}), \nabla] = -\frac{1}{r}\nabla + \frac{1}{r}\frac{x}{r}(\partial_r + \frac{1}{r})$$

Proof.

$$\begin{aligned}
(\partial_r + \frac{1}{r})\partial_j f &= \partial_j(\partial_r + r^{-1})f - r^{-1}(\partial_j - \frac{x_j}{r}\partial_r)f + x_j r^{-3} f \\
&= \partial_j(\partial_r + r^{-1})f - r^{-1}\partial_j f + \frac{x_j}{r} r^{-1}(\partial_r + \frac{1}{r})f.
\end{aligned}$$

$\square$

Note that Lemmas A.5 and A.6 imply

$$\frac{1}{2}[\frac{x}{r}, \Delta] = [(\partial_r + \frac{1}{r}), \nabla]. \tag{A.2}$$

Lemma A.7.

$$[\partial_r, \partial_j] = -r^{-1}(\partial_j - \frac{x_j}{r}\partial_r)$$

Proof.

$$\begin{aligned}
[\partial_r, \partial_j] &= [x_i r^{-1}\partial_i, \partial_j] \\
&= -r^{-1}\partial_j + x_i x_j r^{-3}\partial_i \\
&= -r^{-1}(\partial_j - \frac{x_j}{r}\partial_r)
\end{aligned}$$

$\square$

Lemma A.8.

$$[\partial_i, x_a\partial_b - x_b\partial_a] = \delta_{ia}\partial_b - \delta_{ib}\partial_a$$

Proof.

$$[\partial_i, x_a\partial_b - x_b\partial_a] = [\partial_i, x_a\partial_b] - [\partial_i, x_b\partial_a] = \delta_{ia}\partial_b - \delta_{ib}\partial_a$$

$\square$

Lemma A.9.

$$[\partial_i, r\partial_r] = \partial_i$$

Proof.

$$\begin{aligned}
[\partial_i, r\partial_r] &= [\partial_i, x_j\partial_j] \\
&= \delta_{ij}\partial_j \\
&= \partial_i.
\end{aligned}$$

$\square$

Lemma A.10.

$$[\Omega, S_r] = 0$$

Proof.

$$\begin{aligned}
[x_a\partial_b - x_b\partial_a, x_j\partial_j] &= [x_a\partial_b, x_j\partial_j] - [x_b\partial_a, x_j\partial_j] \\
&= x_a\delta_{bj}\partial_j - x_j\delta_{aj}\partial_b - x_b\delta)aj\partial_j + x_j\delta_{bj}\partial_a \\
&= x_a\partial_b - x_a\partial_b - x_b\partial_a + x_b\partial_a \\
&= 0.
\end{aligned}$$

$\square$

A.2 Detailed Calculations for Proposition 2.4

A.2.1 Establishing (2.22)

In this section we prove (2.22) which states

$$|\tau|^{N+3-j}\|\nabla^j g_{high}\|_{\mathcal{LE}^*} \lesssim \|\nabla^{N+3} g\|_{\mathcal{LE}^*} \tag{A.3}$$

where

$$g_{high} = \int e^{ix\cdot\xi}\hat{g}(\xi)\chi_{>|\tau|}(\xi)\ d\xi$$

and $1 \lesssim |\tau|$.

Set $\beta(\xi) = \chi_{\approx 1}(\xi)$. We calculate

$$\begin{aligned}
\left||\tau|^{N+3-j}\nabla^j g_{high}\right| &= |\tau|^{N+3-j}\left|\int e^{ix\cdot\xi}\chi_{>\tau}(\xi)\xi^j\hat{g}(\xi)\ d\xi\right| \\
&= \sum_{n\geq \ln\tau} \frac{|\tau|^{N+3-j}}{2^{n(N+3-j)}}\left|\int e^{ix\cdot\xi}\frac{2^{n(N+3-j)}}{|\xi|^{N+3}}\beta\left(\frac{\xi}{2^n}\right)\xi^{N+3+j}\hat{g}(\xi)\ d\xi\right| \\
&\lesssim \sum_{n\geq \ln\tau} \frac{|\tau|^{N+3-j}}{2^{n(N+3-j)}}\int \frac{2^{3n}}{(1+2^n|x-y|)^M}\left|\nabla_y^{N+3}g(y)\right|\ dy \qquad \text{for all } M.
\end{aligned}$$

To establish the last inequality, we use the fact that under the inverse Fourier transform, multiplication of operators becomes convolution of the inverse transform of the operators. Setting $\psi(\xi) = \frac{2^{n(N+3-j)}}{|\xi|^{N+3}}\xi^j\beta\left(\frac{\xi}{2^n}\right)$, we can see $|\check{\psi}(x)| = \left|\int e^{ix\cdot\xi}\psi(\xi)\ d\xi\right| \lesssim_m \frac{2^{3n}}{(1+2^n|x|)^M}$ for any M by a change of variables with $\zeta = \frac{\xi}{2^n}$ and integration by parts, along with the observation that $\psi(\xi) \in \mathcal{S}$ where $\mathcal{S}$ denotes the class of Schwartz functions.

Now we have

$$\begin{aligned}
&|\tau|^{N+3-j}\|\nabla^j g_{high}\|_{\mathcal{LE}^*,} \\
&\qquad \lesssim \sum_m 2^{\frac{m}{2}}\sum_{n\geq\ln\tau}\frac{|\tau|^{N+3-j}}{2^{n(N+3-j)}}\sum_k\left\|\int_{|y|\approx 2^k}\frac{2^{3n}}{(1+2^n|x-y|)^M}\nabla_y^{N+3}g(y)\ dy\right\|_{L_x^2(A_m)}.
\end{aligned}$$

We wish to show the right hand side is bounded by $\|\nabla^{N+3}g\|_{\mathcal{LE}^*}$. This requires slightly different arguments for $k < m$, $k > m$, and $k \approx m$.

When $k < m-3$ we have $\left|\frac{2^{3n}}{(1+2^n|x-y|)^M}\right| \lesssim 2^{(3-M)n}2^{-Mm}$. Setting $M=3$ we find

$$\begin{aligned}
&\sum_m 2^{\frac{m}{2}} \sum_{n\geq \ln\tau} \frac{|\tau|^{N+3-j}}{2^{n(N+3-j)}} \sum_{k<m-3} \Big\| \int_{|y|\approx 2^k} \frac{2^{3n}}{(1+2^n|x-y|)^3} \nabla_y^{N+3} g(y)\ dy \Big\|_{L^2_x(A_m)} \\
&\lesssim \sum_m \sum_{n\geq \ln\tau} \frac{|\tau|^{N+3-j}}{2^{n(N+3-j)}} \sum_{k<m-3} 2^{-m} 2^{\frac{3k}{2}} \|\nabla^{N+3} g\|_{L^2(A_k)} \\
&\lesssim \sum_k \sum_{m>k+3} 2^{-m} 2^k 2^{\frac{k}{2}} \|\nabla^{N+3} g\|_{L^2(A_k)} \\
&\lesssim \|\nabla^{N+3} g\|_{\mathcal{LE}^*}.
\end{aligned}$$

When $k > m+3$ we have $\left|\frac{2^{3n}}{(1+2^n|x-y|)^M}\right| \lesssim 2^{(3-M)n}2^{-Mk}$. Setting $M=3$ we find

$$\begin{aligned}
&\sum_m 2^{\frac{m}{2}} \sum_{n\geq \ln\tau} \frac{|\tau|^{N+3-j}}{2^{n(N+3-j)}} \sum_{k>m+3} \Big\| \int_{|y|\approx 2^k} \frac{2^{3n}}{(1+2^n|x-y|)^3} \nabla_y^{N+3} g(y)\ dy \Big\|_{L^2_x(A_m)} \\
&\lesssim \sum_m \sum_{n\geq \ln\tau} \frac{|\tau|^{N+3-j}}{2^{n(N+3-j)}} \sum_{k>m+3} 2^{2m} 2^{-\frac{3k}{2}} \|\nabla_y^{N+3} g(y)\|_{L^2_y(A_k)} \\
&\lesssim \sum_k \sum_{m<k-3} 2^{2m} 2^{-2k} 2^{\frac{k}{2}} \|\nabla^{N+3} g\|_{L^2(A_k)} \\
&\lesssim \|\nabla^{N+3} g\|_{\mathcal{LE}^*}.
\end{aligned}$$

When $k \approx m$ we set $M=4$ and use Young's inequality to find

$$\begin{aligned}
&\sum_m 2^{\frac{m}{2}} \sum_{n\geq \ln\tau} \frac{|\tau|^{N+3-j}}{2^{n(N+3-j)}} \left\| \int_{|y|\approx 2^m} \frac{2^{3n}}{(1+2^n|x-y|)^4} \nabla_y^{N+3} g(y)\ dy \right\|_{L^2_x(A_m)} \\
&\lesssim \sum_m 2^{\frac{m}{2}} \sum_{n\geq \ln\tau} \frac{|\tau|^{N+3-j}}{2^{n(N+3-j)}} \left\| \frac{2^{3n}}{(1+2^n|x|)^4} \right\|_{L^1_x} \|\nabla^{N+3} g\|_{L^2(A_m)} \\
&\lesssim \|\nabla^{N+3} g\|_{\mathcal{LE}^*}.
\end{aligned}$$

This concludes the proof of (A.3).

A.2.2 Establishing (2.24)

In this section we prove (2.24), which states

$$|\tau|^{-j} \|\nabla^{j+k} g_{low}\|_{\mathcal{LE}^*} \lesssim \|\nabla^k g\|_{\mathcal{LE}^*} \tag{A.4}$$

where

$$g_{low} = \int e^{ix\cdot\xi}\hat{g}(\xi)\chi_{<|\tau|}(\xi)\ d\xi$$

and $1 \lesssim |\tau|$.

This is done in a manner similar to how we showed (A.3). Set $\beta(\xi) = \chi_{\approx}(\xi)$. We calculate

$$\begin{aligned}
\left||\tau|^{-j}\nabla^{j+k} g_{low}\right| &= |\tau|^{-j}\left|\int e^{ix\cdot\xi}\chi_{>\tau}(\xi)\xi^{j+k}\hat{g}(\xi)\ d\xi\right| \\
&= \sum_{n\leq \ln\tau} \frac{2^{nj}}{|\tau|^j}\left|\int e^{ix\cdot\xi}\frac{\xi^j}{2^{nj}}\beta\left(\frac{\xi}{2^n}\right)\xi^k\hat{g}(\xi)\ d\xi\right| \\
&\lesssim \sum_{n\leq\ln\tau}\frac{2^{nj}}{|\tau|^j}\int\frac{2^{3n}}{(1+2^n|x-y|)^M}\left|\nabla^k g(y)\right|\ dy \qquad \text{for any } M.
\end{aligned}$$

Now we have

$$\begin{aligned}
&|\tau|^{-j}\|\nabla^{j+k}g_{low}\|_{\mathcal{LE}^*} \\
&\quad\lesssim \sum_m 2^{\frac{m}{2}}\sum_{n\leq\ln\tau}\frac{2^{nj}}{|\tau|^j}\sum_\ell\left\|\int_{|y|\approx 2^\ell}\frac{2^{3n}}{(1+2^n|x-y|)^M}\left|\nabla^k g(y)\right|\ dy\right\|_{L^2_x(A_m)}.
\end{aligned}$$

When $\ell < m-3$ we have $\left|\frac{2^{3n}}{(1+2^n|x-y|)^M}\right| \lesssim 2^{(3-M)n}2^{-Mm}$. Setting $M=4$ we find

$$\begin{aligned}
&\sum_m 2^{\frac{m}{2}}\sum_{n\leq\ln\tau}\frac{2^{nj}}{|\tau|^j}\sum_{\ell<m-3}\left\|\int_{|y|\approx 2^\ell}\frac{2^{3n}}{(1+2^n|x-y|)^4}\nabla^k g(y)\ dy\right\|_{L^2_x(A_m)} \\
&\quad\lesssim \sum_m\sum_{n\leq\ln\tau}\frac{2^{n(j-1)}}{|\tau|^j}\sum_{\ell<m-3}2^{-2m}2^{\frac{3\ell}{2}}\|\nabla^k g\|_{L^2(A_\ell)} \\
&\quad\lesssim \sum_\ell\sum_{m>\ell+3}|\tau|^{-1}\ln|\tau|2^{-2m}2^\ell 2^{\frac{\ell}{2}}\|\nabla^k g\|_{L^2(A_\ell)} \\
&\quad\lesssim \|\nabla^k g\|_{\mathcal{LE}^*}.
\end{aligned}$$

When $\ell > m+3$ we have $\left|\frac{2^{3n}}{(1+2^n|x-y|)^M}\right| \lesssim 2^{(3-M)n}2^{-M\ell}$. Setting $M=4$ we find

$$\begin{aligned}
&\sum_m 2^{\frac{m}{2}} \sum_{n\le \ln\tau} \frac{2^{nj}}{|\tau|^j} \sum_{\ell > m+3} \Big\| \int_{|y|\approx 2^\ell} \frac{2^{3n}}{(1+2^n|x-y|)^4} \nabla^k g(y)\ dy \Big\|_{L^2_x(A_m)} \\
&\lesssim \sum_m \sum_{n\le\ln\tau} \frac{2^{n(j-1)}}{|\tau|^j} \sum_{\ell>m+3} 2^{2m} 2^{-\frac{5\ell}{2}} \|\nabla^k g\|_{L^2(A_\ell)} \\
&\lesssim \sum_\ell \sum_{m<\ell-3} |\tau|^{-1} \ln\tau 2^{2m} 2^{-3\ell} 2^{\frac{\ell}{2}} \|\nabla^k g\|_{L^2(A_\ell)} \\
&\lesssim \|\nabla^k g\|_{\mathcal{LE}^*}.
\end{aligned}$$

When $\ell \approx m$ we set $M=4$ and use Young's inequality to find

$$\begin{aligned}
&\sum_m 2^{\frac{m}{2}} \sum_{n\le\ln\tau} \frac{2^{nj}}{|\tau|^j} \Big\| \int_{|y|\approx 2^m} \frac{2^{3n}}{(1+2^n|x-y|)^4} \nabla^k g(y)\ dy \Big\|_{L^2_x(A_m)} \\
&\lesssim \sum_m 2^{\frac{m}{2}} \sum_{n\le\ln\tau} \frac{2^{nj}}{|\tau|^j} \Big\| \frac{2^{3n}}{(1+2^n|x|)^4} \Big\|_{L^1_x} \|\nabla^k g\|_{L^2(A_m)} \\
&\lesssim \|\nabla^k g\|_{\mathcal{LE}^*}.
\end{aligned}$$

This concludes the proof of (A.4).

A.3 Chapter 3 Intro: Restatements of Assumption 3

Here we justify the restatements of Assumption 3 at the beginning of Chapter 3. Recall

$$\begin{aligned}
&\mathfrak{g} = \mathfrak{m} + \mathfrak{f} + \mathfrak{h} \\
&\mathfrak{f} = \mathfrak{f}_{tt} dt^2 + \mathfrak{f}_{ti} dt dx_i + \mathfrak{f}_{ij} dx_i dx_j \\
&\mathfrak{h} = \mathfrak{h}_{tt} dt^2 + \mathfrak{h}_{tr} dt dr + \mathfrak{h}_{rr} dr^2 + r^2 \mathfrak{h}_{\omega\omega} d\omega^2 + r^2 \sin^2\theta \mathfrak{h}_{\omega\omega} d\phi^2
\end{aligned}$$

where $\mathfrak{f}_{\alpha\beta} \in \ell^1 S(r^{-k})$ for $\alpha, \beta \in \{t, x_1, x_2, x_3\}$, and $\mathfrak{h}_{\gamma\delta} \in S_{rad}(r^{-k})$ for $\gamma, \delta \in \{t, r, \theta, \phi\}$. As in Chapter 3, we use α, β to indicate rectangular coordinates and γ, δ to indicate spherical coordinates.

We begin by converting $\mathfrak{m}$ and $\mathfrak{f}$ to spherical coordinates. Rectangular and spherical coordinates are related by the equations

$$t = t, \quad x_1 = r\sin\theta\sin\phi, \quad x_2 = r\sin\theta\cos\phi, \quad x_3 = r\cos\theta$$

which yields

$$\frac{\partial t}{\partial t} = 1, \quad \frac{\partial x_1}{\partial r} = \sin\theta\sin\phi, \quad \frac{\partial x_1}{\partial \theta} = r\cos\theta\sin\phi, \quad \frac{\partial x_1}{\partial \phi} = r\sin\theta\cos\phi$$

$$\frac{\partial x_2}{\partial r} = \sin\theta\cos\phi, \quad \frac{\partial x_2}{\partial \theta} = r\cos\theta\cos\phi, \quad \frac{\partial x_2}{\partial \phi} = -r\sin\theta\sin\phi$$

$$\frac{\partial x_3}{\partial r} = \cos\theta, \quad \frac{\partial x_3}{\partial \theta} = -r\sin\theta$$

and all other partial derivatives are 0.

The Minkowski metric $\mathfrak{m}$ in spherical coordinates is given by

$$\mathfrak{m} = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2.$$

This is a well known fact, which we simply state.

To find $\mathfrak{f}$ in spherical coordinates we use

$$\mathfrak{f}_{\gamma\delta} = \frac{\partial\alpha}{\partial\gamma}\frac{\partial\beta}{\partial\gamma}\mathfrak{f}_{\alpha\beta}.$$

This yeilds

$$\mathfrak{f}_{tt}, \mathfrak{f}_{tr}, \mathfrak{f}_{rr} \in \ell^1 S(r^{-k}); \quad \mathfrak{f}_{t\theta}, \mathfrak{f}_{r\theta} \in r\ell^1 S(r^{-k}); \quad \mathfrak{f}_{\theta\theta} \in r^2\ell^1 S(r^{-k})$$

$$\mathfrak{f}_{t\phi}, \mathfrak{f}_{r\phi} \in r\sin\theta\ell^1 S(r^{-k}); \quad \mathfrak{f}_{\theta\phi}, \mathfrak{f}_{\phi\phi} \in r^2\sin^2\theta\ell^1 S(r^{-k})$$

Relabeling, we can write

$$\left[\mathfrak{f}^{\gamma\delta}\right] = \begin{bmatrix} \mathfrak{f}_{tt} & \mathfrak{f}_{tr} & r\mathfrak{f}_{t\theta} & r\sin\theta\mathfrak{f}_{t\phi} \\ \mathfrak{f}_{tr} & \mathfrak{f}_{rr} & r\mathfrak{f}_{r\theta} & r\sin\theta\mathfrak{f}_{r\phi} \\ r\mathfrak{f}_{t\theta} & r\mathfrak{f}_{r\theta} & r^2\mathfrak{f}_{\theta\theta} & r^2\sin\theta\mathfrak{f}_{\phi\phi} \\ r\sin\theta\mathfrak{f}_{t\phi} & r\sin\theta\mathfrak{f}_{r\phi} & r^2\sin\theta\mathfrak{f}_{\theta\phi} & r^2\sin^2\theta\mathfrak{f}_{\phi\phi} \end{bmatrix}$$

where $\mathfrak{f}_{\gamma\delta} \in \ell^1 S(r^{-\kappa})$. This concludes the calculations establishing Assumption 3 in spherical coordinates.

It follows from the above calculations that the coefficient matrix for $\mathfrak{g}$ in spherical coordinates is given by

$$\begin{bmatrix} -1+\mathfrak{f}_{tt}+\mathfrak{h}_{tt} & \mathfrak{f}_{tr}+\mathfrak{h}_{tr} & r\mathfrak{f}_{t\theta} & r\sin\theta\mathfrak{f}_{t\phi} \\ \mathfrak{f}_{tr}+\mathfrak{h}_{tr} & 1+\mathfrak{f}_{rr}+\mathfrak{h}_{rr} & r\mathfrak{f}_{r\theta} & r\sin\theta\mathfrak{f}_{r\phi} \\ r\mathfrak{f}_{t\theta} & r\mathfrak{f}_{r\theta} & r^2(1+\mathfrak{f}_{\theta\theta}+\mathfrak{h}_{\omega\omega}) & r^2\sin\theta\mathfrak{f}_{\theta\phi} \\ r\sin\theta\mathfrak{f}_{t\phi} & r\sin\theta\mathfrak{f}_{r\phi} & r^2\sin\theta\mathfrak{f}_{\theta\phi} & r^2\sin^2\theta(1+\mathfrak{f}_{\phi\phi}+\mathfrak{h}_{\omega\omega}) \end{bmatrix}$$

where $\mathfrak{f}_{\gamma\delta} \in \ell^1 S(r^{-\kappa})$ and $\mathfrak{h}_{\gamma\delta} \in \ell^1 S(r^{-\kappa})$.

We now consider the inverse $\mathfrak{g}^{\gamma\delta}$, which can be found by multiplying the matrix of minors by $1/\det \mathfrak{g}_{\gamma\delta}$. The determinant of $\mathfrak{g}_{\gamma\delta}$ is the sum of terms each of which can be written as $\mathfrak{g}_1 \cdot \mathfrak{g}_2 \cdot \mathfrak{g}_3 \cdot \mathfrak{g}_4$ where no two of the $\mathfrak{g}_i$ are in the same row or column. To see what symbol class the determinant belongs to, we first note that any such combination will necessarily have a $r^4 \sin^2\theta$ factor. Pulling this out we can examine the above matrix without the r and $\sin\theta$ factors. Any term resulting from multiplication of at least two decaying terms can will be in $\ell^1 S(r^{-\kappa})$, so we are really concerned with terms resulting from the multiplication of at least 3 constants.This only occurs when multiplying along the diagonal. Multiplying along the diagonal (ignoring the r^2 and $\sin^2\theta$ factors) yields

$$-1+\rho_\ell^{-\kappa}+\rho_r^{-\kappa}.$$

Thus $\det\mathfrak{g}_{\gamma\delta}$ can be written as

$$\det\mathfrak{g}_{\gamma\delta} = r^4\sin^2\theta(-1+\rho_\ell^{-\kappa}+\rho_r^{-\kappa}).$$

In rectangular coordinates we have

$$\det\mathfrak{g}_{\alpha\beta} = \frac{\det\mathfrak{g}_{\gamma\delta}}{r^4\sin^2\theta}$$

Since our metric is non-degenerate by assumption and $\det\mathfrak{g}_{\alpha\beta} \to -1$ we have $|\det\mathfrak{g}_{\alpha\beta}| \gtrsim 1$.

We use the same argument for the matrix of minors, which we denote $\mathfrak{g}^*_{\gamma\delta}$. Each element $\mathfrak{g}^*_{\gamma\delta}$ is given by the determinant of the 3×3 submatrix found after removing row α and column β. The terms with the least decay from from multiplying two elements with a $1+\rho_\ell^{-\kappa}+\rho_r^{-\kappa}$ factor. We find for

$$\begin{bmatrix} r^4\sin^2\theta(-1+\mathfrak{f}^{tt}+\mathfrak{h}^{tt}) & r^4\sin^2\theta(\mathfrak{f}^{tr}+\mathfrak{h}^{tr}) & r^3\sin^2\theta\mathfrak{f}^{t\theta} & r^3\sin\theta\mathfrak{f}^{t\phi} \\ r^4\sin^2\theta(\mathfrak{f}^{tr}+\mathfrak{h}^{tr}) & r^4\sin^2\theta(1+\mathfrak{f}^{rr}+\mathfrak{h}^{rr}) & r^3\sin^2\theta\mathfrak{f}_{r\theta} & r^3\sin\theta\mathfrak{f}^{r\phi} \\ r^3\sin^2\theta\mathfrak{f}_{t\theta} & r^3\sin^2\theta\mathfrak{f}^{r\theta} & r^2\sin^2\theta(1+\mathfrak{f}^{\theta\theta}+\mathfrak{h}^{\omega\omega}) & r^2\sin\theta\mathfrak{f}^{\theta\phi} \\ r^3\sin\theta\mathfrak{f}^{t\phi} & r^3\sin\theta\mathfrak{f}^{r\phi} & r^2\sin\theta\mathfrak{f}^{\theta\phi} & r^2(1+\mathfrak{f}^{\phi\phi}+\mathfrak{h}^{\omega\omega}) \end{bmatrix}$$

where $\mathfrak{f}^{\gamma\delta} \in \ell^1 S(r^{-\kappa})$ and $\mathfrak{h}^{\gamma\delta} \in \ell^1 S(r^{-\kappa})$. Thus dividing by the determinant

$$\det\mathfrak{g}_{\gamma\delta} = r^4\sin^2\theta(-1+\rho_\ell^{-\kappa}+\rho_r^{-\kappa})$$

yields the desired form for $\mathfrak{g}^{\gamma\delta}$ stated in Chapter 3.

A.4 Establishing (3.19)

Here we show

$$\sum_{i,j=1}^{3}(\partial_r(\mathfrak{h}^{tt}+\mathfrak{h}^{\omega\omega}))(\delta_{ij}x_i r^{-1}\partial_j - x_i^2 x_j r^{-3}\partial_j) + (\mathfrak{h}^{tt}+\mathfrak{h}^{\omega\omega})\partial_i(\delta_{ij}-x_ix_jr^{-2})\partial_j = (\mathfrak{h}^{tt}+\mathfrak{h}^{\omega\omega})r^{-2}\Delta_\omega.$$

Summing over i and j, the first term vanishes since

$$\sum_{ij}\delta_{ij}x_ir^{-1}\partial_j - x_i^2x_jr^{-3}\partial_j = \sum_j x_jr^{-1}\partial_j - x_jr^{-1}\partial_j.$$

For the second term, we recall Δ_ω is defined by

$$\Delta = \partial_r^2 + \frac{2}{r}\partial_r + r^{-2}\Delta_\omega.$$

The fact that

$$\partial_i(\delta_{ij}-x_ix_jr^{-2})\partial_j = r^{-2}\Delta_\omega$$

can be seen by by noting that the Laplacian is given by $\Box_L$ where

$$L^{\gamma\delta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{bmatrix}$$

So the angular part of the Laplacian is given by $\Box_{L_\omega}$ where

$$L_\omega^{\gamma\delta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2 \sin^2\theta} \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and in standard coordinates we have

$$L_\omega^{\alpha\beta} = \delta_{ij} - x_i x_j r^{-2}.$$

Thus

$$r^{-2}\Delta_\omega = \partial_i(\delta_{ij} - x_i x_j r^{-2})\partial_j.$$

Alternatively the fact that

$$\partial_i(\delta_{ij} - x_i x_j r^{-2})\partial_j = r^{-2}\Delta_\omega$$

can be seen by direct calculation. We have

$$\partial_r = x_j r^{-1}\partial_j$$

$$\begin{aligned} \partial_r^2 &= x_j r^{-2}\partial_j - x_j r^{-2}\partial_j + x_i x_j r^{-2}\partial_i\partial_j \\ &= x_i x_j r^{-2}\partial_i\partial_j. \end{aligned}$$

We calculate

$$\begin{aligned} \partial_i(\delta_{ij} - x_i x_j r^{-2})\partial_j &= \Delta - \partial_i(x_i x_j r^{-2})\partial_j \\ &= \Delta - x_i x_j r^{-2}\partial_i\partial_j - \partial_i(x_i x_j r^{-2})\partial_j \\ &= \Delta - \partial_r^2 - 3x_j r^{-2}\partial_j - \delta_{ij} x_i r^{-2}\partial_j + 2x_i^2 x_j r^{-3}\partial_j \\ &= \Delta - \partial_r^2 - 2x_j r^{-2}\partial_j \\ &= \Delta - \partial_r^2 - 2r^{-1}\partial_r \\ &= r^{-2}\Delta_\omega. \end{aligned}$$

A.5 Detailed Calculations for Lemma 4.1

A.5.1 Bounds on Coefficients of the Expansion

We first show that

$$c_j = \sum_m \int g_m(y) y^j \; dy, \qquad e_j(r) = \sum_m r^{\lambda-1-j}(-\chi_{<m+2}) \int g_m(y) y^j \; dy$$

$$d(r) = \sum \chi_{>m+2}(r) \int g_m(y) y^{\lambda-1} \; dy$$

with $g_m = \beta_{\approx m} g$ satisfy

$$\sum_{j=0}^{\lambda-2} |c_j| + \|e_j\|_{\ell^1 S(1)} + \|d\|_{L^\infty} + \|S_r d\|_{\ell^1 S(1)} \lesssim \|\langle r\rangle^\lambda g\|_{\mathcal{LE}^*}.$$

Our calculations for c_j and e_j take advantage of the assumption $0 \leq j \leq \lambda - 2$. In each calculation we use

$$\|\langle r\rangle^p g\|_{L^1(A_m)} \lesssim \|\langle r\rangle^{p+\frac{3}{2}} g\|_{L^2(A_m)}.$$

For c_j we find

$$\begin{aligned} |c_j| &\lesssim \sum_m \|g y^j\|_{L^1(A_m)} \\ &\lesssim \sum_m \|\langle r\rangle^{j+3/2} g\|_{L^2(A_m)} \\ &= \|\langle r\rangle^{j+1} g\|_{\mathcal{LE}^*} \\ &\leq \|\langle r\rangle^\lambda g\|_{\mathcal{LE}^*}. \end{aligned}$$

An identical calculation shows $\|d\|_{L^\infty} \lesssim \|\langle r\rangle^\lambda g\|_{\mathcal{LE}^*}$. For $S_r d$ we find

$$\partial_r d = \sum_m 2^{-m-2} \chi_{\approx m+2}(r) \int g_m(y) y^{\lambda-1} \; dy$$

so that

$$\sum_\ell \|S_r d\|_{L^\infty(A_\ell)} \leq \sum_m \|g_m y^{\lambda-1}\|_{L^1(A_m)} \lesssim \|\langle r\rangle^\lambda g\|_{\mathcal{LE}^*}.$$

Similarly, higher order derivatives yield $\partial_r^n d \approx r^{-n+1} \partial_r d$ so that indeed we have $\|S_r d\|_{\ell^1 S(1)} \lesssim$

$\|\langle r\rangle^{\lambda} g\|_{\mathcal{LE}^*}$, as desired.

For e_j we find

$$\begin{aligned}\sum_\ell \|e_j\|_{L^\infty(A_\ell)} &\lesssim \sum_\ell \sum_{m>\ell-2} 2^{\ell(\lambda-1-j)} \int |g_m(y) y^j| \ dy \\ &\lesssim \sum_m \sum_{\ell<m+2} 2^{(\ell-m)(\lambda-1-j)} \|\langle r\rangle^{\lambda-1} g\|_{L^1(A_m)} \\ &\lesssim \sum_m \|\langle r\rangle^{\lambda+\frac{1}{2}} g\|_{L^2(A_m)} \sum_{\ell<m+2} 2^{(\ell-m)} \\ &\lesssim \|\langle r\rangle^{\lambda} g\|_{\mathcal{LE}^*}.\end{aligned}$$

For higher derivatives of e_j we note

$$\partial_r e_j = \Big(\sum_m 2^{-m-2} \chi_{\approx m-2}(r) \langle r\rangle^{\ell-1-j} + (\ell-1-j)\chi_{<m+2}(r)\langle r\rangle^{\ell-2-j}\Big) \int g_m(y) y^j \ dy.$$

Thus higher order derivatives of e_j come with increased decay and $\|e_j\|_{\ell^1 S(1)} \lesssim \|\langle r\rangle^{\lambda} g\|_{\mathcal{LE}^*}$, as desired.

A.5.2 Proving (4.20), (4.21) , and (4.22)

To prove (4.20) we calculate

$$\Delta(c_j \nabla^j \langle r\rangle^{-1}) = c_j \nabla^j \Delta \langle r\rangle^{-1}.$$

Since $\langle r\rangle = r$ for $r \geq 2$ and $\langle r\rangle \equiv 1$ for $r \leq 1$ we see $\Delta\langle r\rangle^{-1} = 0$ for $r \leq 1$ and $r \geq 2$. Thus $\Delta\langle r\rangle^{-1}$ is compactly supported on $[1, 2]$ and we easily find

$$\|c_j \nabla^j \Delta \langle r\rangle^{-1}\|_{Z^{n,\lambda}} \lesssim |c_j| \lesssim \|\langle r\rangle^{\lambda} g\|_{\mathcal{LE}^*} \lesssim \|g\|_{Z^{0,\lambda}}$$

using (4.4).

To prove (4.21) we write

$$e_j(\nabla^j \langle r\rangle^{-1})\langle r\rangle^{j-\lambda+1} = e_j \rho^{-\lambda}$$

where $\rho^{-\lambda} \in S(r^{-\lambda})$. Therefore

$$\Delta(e_j(\nabla^j \langle r\rangle^{-1})\langle r\rangle^{j-\lambda+1}) = (\Delta e_j)\rho^{-\lambda} + 2\nabla e_j \cdot \nabla \rho^{-\lambda} + e_j(\Delta \rho^{-\lambda})$$

and we find using Lemma A.3 and (4.4)

$$\begin{aligned}
&\|\Delta(e_j(\nabla^j \langle r\rangle^{-1})\langle r\rangle^{j-\lambda+1})\|_{Z^{n,\lambda}} \\
&\quad \lesssim \sum_{|\alpha|=2} \|(\partial^\alpha e_j)\rho^{-\lambda}\|_{Z^{n,\lambda}} + \sum_{|\alpha|=1} \|(\partial^\alpha e_j)(\partial^\alpha \rho^{-\lambda})\|_{Z^{n,\lambda}} + \|e_j \rho^{-\lambda-2}\|_{Z^{n,\lambda}} \\
&\quad \lesssim \|e_j\|_{\ell^1 S(1)} \\
&\quad \lesssim \|g\|_{Z^{0,\lambda}}.
\end{aligned}$$

To prove (4.22) we calculate

$$\begin{aligned}
&\Delta(d(r)\nabla^{\lambda-1}\langle r\rangle^{-1}) \\
&\quad = (\Delta d(r))\nabla^{\lambda-1}\langle r\rangle^{-1} + 2\nabla d(r) \cdot \nabla^{\lambda}\langle r\rangle^{-1} + d(r)\nabla^{\lambda-1}(\Delta\langle r\rangle^{-1}) \\
&\quad = \Big(r^{-1}\partial_r(S_r d) + r^{-2}(S_r d)\Big)\nabla^{\lambda-1}\langle r\rangle^{-1} + r^{-1}S_r d\frac{x}{r} \cdot \nabla^{\lambda}\langle r\rangle^{-1} + d(r)\nabla^{\lambda-1}(\Delta\langle r\rangle^{-1}) \\
&\quad = \partial_r(S_r d)\rho^{-\lambda-1} + (S_r d)\rho^{-\lambda-2} + d(r)\nabla^{\lambda-1}(\Delta\langle r\rangle^{-1})
\end{aligned}$$

where $\rho^{-\lambda-1} \in S(r^{-\lambda-1})$ and $\rho^{-\lambda-2} \in S(r^{-\lambda-2})$. For the first two terms we find

$$\|\partial_r(S_r d)\rho^{-\lambda-1}\|_{Z^{n,\lambda}} + \|(S_r d)\rho^{-\lambda-2}\|_{Z^{n,\lambda}} \lesssim \|S_r d\|_{\ell^1 S(1)} \lesssim \|g\|_{Z^{0,\lambda}}$$

using Lemma A.3 and (4.4) as we did to prove (4.21). For the last term we find

$$\|d(r)\nabla^{\lambda-1}(\Delta\langle r\rangle^{-1})\|_{Z^{n,\lambda}} \lesssim \|d\|_{L^\infty} \lesssim \|g\|_{Z^{0,\lambda}}$$

using the fact that $\Delta\langle r\rangle^{-1}$ is compactly supported and (4.4).

A.6 Detailed Calculations for Proposition 4.5

A.6.1 Establishing Equation (4.53)

In this section we provide the calculations to show

$$P_\tau(\phi e^{-i\tau\langle r\rangle}) = \big[(\Delta + P^2)\phi\big]e^{-i\tau\langle r\rangle} + 2\tau\big[(\partial_r + \frac{1}{r})\phi\big]e^{-i\tau\langle r\rangle} + \Big[\big(\tau(\rho_\ell^{-\kappa-1} + \rho_\ell^{-\kappa}\nabla) + \tau^2\rho_\ell^{-\kappa}\big)\phi\Big]e^{-i\tau\langle r\rangle}, \tag{A.5}$$

which is equation (4.53) in Proposition 4.5. Here we use the notation $\rho_\ell^q \in \ell^1 S(r^q)$.

Recall

$$P_\tau = \Delta + \tau^2 + i\tau P^1 + P^2$$

where

$$P^1 = \partial_i p_1^i + p_1^i\partial_i, \qquad p_1^i \in \ell^1 S(r^{-\kappa})$$

and

$$P^2 = \partial_i p_2^{ij}\partial_j + p_2^\omega \Delta_\omega + V_\ell + V_r,$$

$$p_2^{ij} \in \ell^1 S(r^{-\kappa}), \quad p_2^\omega, V_r \in S_{rad}(r^{-\kappa-2}), \quad \text{and} \quad V_\ell \in \ell^1 S(r^{-\kappa-2}).$$

First we calculate $\Delta(\phi e^{-i\tau\langle r\rangle})$:

$$\begin{aligned}
\Delta(\phi e^{-i\tau\langle r\rangle}) &= (\Delta\phi)e^{-i\tau\langle r\rangle} + \phi(\Delta e^{-i\tau\langle r\rangle}) + 2(\nabla\phi \cdot \nabla e^{-i\tau\langle r\rangle}) \\
&= (\Delta\phi)e^{-i\tau\langle r\rangle} + \phi\Big(-\tau^2 e^{-i\tau\langle r\rangle} - 2\frac{i\tau}{r}e^{-i\tau\langle r\rangle}\Big) - 2\frac{i\tau x_i}{r}(\partial_i\phi)e^{-i\tau\langle r\rangle} \\
&= (\Delta\phi)e^{-i\tau\langle r\rangle} - 2i\tau\big[(\partial_r + \frac{1}{r})\phi\big]e^{-i\tau\langle r\rangle} - \tau^2\phi e^{-i\tau\langle r\rangle}.
\end{aligned}$$

Thus we have

$$(\Delta + \tau^2)(\phi e^{-i\tau\langle r\rangle}) = (\Delta\phi)e^{-i\tau\langle r\rangle} - 2i\tau(\partial_r + \frac{1}{r})(\phi)e^{-i\tau\langle r\rangle}. \tag{A.6}$$

Next we calculate $i\tau P^1\phi e^{-i\tau\langle r\rangle}$.

$$\begin{aligned}
i\tau P^1\phi e^{-i\tau\langle r\rangle} &= i\tau\Big(2p_1^i\partial_i(\phi e^{-i\tau\langle r\rangle}) + (\partial_i p_1^i)\phi e^{-i\tau\langle r\rangle}\Big) \\
&= 2p_1^i\frac{\tau^2 x_i}{r}\phi e^{-i\tau\langle r\rangle} + 2i\tau p_1^i(\partial_i\phi)e^{-i\tau\langle r\rangle} + i\tau(\partial_i p_1^i)\phi e^{-i\tau\langle r\rangle} \qquad (A.7) \\
&= \tau^2\rho_\ell^{-\kappa}\phi e^{-i\tau\langle r\rangle} + \tau\rho_\ell^{-\kappa}(\nabla\phi)e^{-i\tau\langle r\rangle} + \tau\rho_\ell^{-\kappa-1}\phi e^{-i\tau\langle r\rangle}.
\end{aligned}$$

Finally we calculate $P^2\phi e^{-i\tau\langle r\rangle}$. Note p_2^{ij} is symmetric, so we have

$$\begin{aligned} P^2\phi e^{-i\tau\langle r\rangle} &= p_2^{ij}\partial_i\partial_j(\phi e^{-i\tau\langle r\rangle}) + (\partial_i p_2^{ij})\partial_j(\phi e^{-i\tau\langle r\rangle}) + p_2^{\omega}\Delta_\omega(\phi e^{-i\tau\langle r\rangle}) + (V_\ell + V_r)\phi e^{-i\tau\langle r\rangle} \\ &= (P^2\phi)e^{-i\tau\langle r\rangle} + 2p_2^{ij}(\partial_i\phi)(\partial_j e^{-i\tau\langle r\rangle}) + p_2^{ij}\phi(\partial_i\partial_j e^{-i\tau\langle r\rangle}) + (\partial_i p_2^{ij})\phi(\partial_j e^{-i\tau\langle r\rangle}) \\ &= (P^2\phi)e^{-i\tau\langle r\rangle} + \tau\rho_\ell^{-\kappa}(\partial_i\phi)e^{-i\tau\langle r\rangle} + \tau^2\rho_\ell^{-\kappa}\phi e^{-i\tau\langle r\rangle} + \tau\rho_\ell^{-\kappa-1}\phi e^{-i\tau\langle r\rangle}. \end{aligned} \tag{A.8}$$

Combining (A.6), (A.7), and (A.8) yields (A.5), as desired.

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